Cosmology 1

2020/2021
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Second intermediate test
Topic: FRW models.
Deadline: May 27, 11:00.

According to Olber's paradox, if the universe is infinite and eternal (and transparent) the night sky should be as bright as the surface of the sun (that is an average star). Indeed, every line of sight will sooner or later touch the surface of a star, and surface brightness does not depend on distance.

(1) Suppose now that the universe is static and eternal, filled with galaxies like the Milky Way, with number density of 10^{-2} Mpc^{-3}, each containing 10^{11} stars like the sun ($R_\odot = 696000$ km) in a disc of radius 10 kpc and negligible width. Assuming that the orientation of such galaxies is random, compute the average length of a photon path first from us to a galaxy, then from us to the surface of a star, and compare both lengths to the horizon size c/H. This should demonstrate that the presence of a cosmological horizon is more than sufficient to solve Olber's paradox.

If you are confused by orientations, assume that all galaxies are face-on, the order of magnitude will be the same. But averaging over orientations is simple.

Now let's compute the number of galaxies one expects to see on the sky in an expanding Universe. For this exercise we fix the Hubble constant to the fiducial value of $H = 70$ km s^{-1} Mpc^{-1}. We consider two cosmological models, an open matter-dominated universe with $\Omega_m = 0.3$ and a flat universe with matter and a cosmological constant, again with $\Omega_m = 0.3$.

(2) As a first step, let's assume that all galaxies were born at redshift $z = 5$, and all of them are visible. We want to compute the number surface density of galaxies, i.e. how many galaxies per square degree are visible on the sky, and what is the fraction of the sky that is covered by a galaxy. Let's proceed as follows:

(a) Compute the comoving distance $d_c(z)$; recall that the integral can be reformulated as:

$$\int_t^{t_0} \frac{cdt}{a(t)} = \int_0^z \frac{cdz}{H(z)}$$

Compute the angular diameter distance $d_d(z)$, and plot both distances as a function of z for the two fiducial cosmologies defined above. Show z from 0 to 5.

(b) Work out the comoving volume of a shell at redshift z and width dz, $dV_c = dV_c/dz \times dz$.

(c) Then compute the number of galaxies visible in one square degree of sky as the integral in z of the number visible at each redshift bin, with z from 0 to 5, and report the values for the two models.

(d) For each redshift bin, compute the fraction of sky covered by galaxies, and integrate it in redshift for the two models. Do we expect that all galaxy images will overlap?

(3) This point is reserved to those that have smoothly gone through the previous calculations. We have made two strong assumptions, that all galaxies are the same and that they are all visible. To relax these assumptions we need to assume a luminosity function for the galaxies. We introduce here a couple of astrophysical concepts.

Apparent magnitude: it quantifies the measured flux f of a sky source, as

$$m = -2.5 \log_{10} f + \text{const}$$

. In this test we do not need to know the constant.

Absolute magnitude: the absolute magnitude M of a sky object is defined as the apparent magnitude it would have if its luminosity distance were 10 pc. One can then relate the apparent and absolute magnitudes as:

$$M - m = -5 \log_{10} d_L - 25$$

where the luminosity distance is measured in Mpc.[1]

As for the luminosity function assume that galaxy luminosities are randomly distributed in absolute magnitude, with a flat distribution from a very faint magnitude to $M = -20.5$. The number density of galaxies in the range from $M = -19.5$ to $M = -20.5$ is $\Phi_* = 10^{-2}$ Mpc^{-3}. Suppose then that we observe down to an apparent magnitude m, that corresponds to an absolute magnitude $M(z)$ at a given z: it is easy to see that the number density of visible galaxies is simply $\Phi_*(M(z) + 20.5)$ if $M(z) > -20.5$, 0 otherwise.

Compute now the "galaxy number counts", the number of galaxies visible on the sky as a function of their magnitude.

(a) First compute the luminosity distance d_L of the two models, and show them along with the other distances.

(b) Then compute and show, as a function of apparent magnitude, what is the highest redshift at which a galaxy of absolute magnitude M_* can be seen. Number counts will have no contribution from higher redshifts.

(c) Finally compute, at each redshift, the number of galaxies that can be seen at a given apparent magnitude, and report the two curves of the number of galaxies as a function of magnitude, in the apparent magnitude range from 12 to 30.

(4) A question for all groups: would this method of number counts be useful to constrain cosmological parameters? Try to elaborate on this, identifying the assumptions that could be wrong here. If you have spare time try to experiment with different parameters to see how number counts change. The bravest may also try to use a more realistic Schechter luminosity function in place of our idealized one.

[1] The right hand term of this equation is known as (minus) the distance modulus $\mu = 5 \log_{10} d_L + 25$.

Solution

(1) We compute the average length of the line of sight to a galaxy as a mean free path,

$$l_G = \frac{1}{n_G \sigma_G}$$

where σ_G is the galaxy cross section and $n_G = 10^{-2}$ Mpc^{-3}. If the angle θ between the normal to the galaxy disc and the line of sight is randomly distributed between $-\pi/2$ and $\pi/2$, the average value of $\cos\theta$ is $2/\pi$, so the average cross section of a galaxy is $\sigma_G = 2R_G^2$ ($R_G = 0.01$ Mpc). The resulting length is:

$$l_G = \frac{1}{2R_G^2 n_G} = 500,000 \text{ Mpc}$$

.

This number alone is much larger than the horizon size $c/H_0 = 4290$ Mpc, and this implies that most sightline will not intersect a galaxy in the presence of such a horizon. However, the cross section of stars in a galaxy can be computed as the cross section of the galaxy multiplied by $N_\star (R_\odot/R_G)^2 \sim 5 \times 10^{-13}$, so to obtain the line of sight to the nearest stars l_G must be divided by this number, obtaining:

$$l_\star = \frac{l_G}{N_\star (R_\odot/R_G)^2} \simeq 10^{18} Mpc$$

This huge number gives an idea on how wide the Universe is with respect to the astrophysical object it contains.

(2a) The next step is to compute distances, starting from the comoving distance. The simplest thing is to numerically integrate $c/H(z)$ in z from $z = 0$ to 5, where

$$H(z) = H_0 \sqrt{\Omega_m (1+z)^3 + \Omega_k (1+z)^2 + \Omega_\Lambda} = H_0 E(z)$$

Figure 1 reports the comoving and angular diameter distances for the two fiducial cosmologies.

(2b) For the flat model, the comoving volume $V_c = 4\pi d_c(z)^3/3$ can be easily differentiated, to obtain $dV_c = 4\pi d_c^2 dz\, c/H(z)$. Then, taking into account that angles are conserved during expansion, and that one square degree corresponds to $(\pi/180)^2$ steradians, the volume subtended by a square degree is:

$$\frac{dV_c}{dz} = \left(\frac{\pi}{180}\right)^2 d_c(z)^2 \frac{c}{H(z)}$$

In the open model, calling $\mathcal{R}_0 = c/H_0\sqrt{\Omega_k}$, the universe curvature radius and $\chi = d_c/\mathcal{R}_0$, the volume is given by $V(\chi) = 2\pi \mathcal{R}_0^3 (\sinh(2\chi)/2 - \chi)$. Differentiating this expression with respect to z, and multiplying the result by $(\pi/180)^2/4\pi$ to pass from full sky to one square degree, we have:

$$\frac{dV_c}{dz} = \frac{1}{2}\left(\frac{\pi}{180}\right)^2 \mathcal{R}_0^3 \frac{\sqrt{\Omega_k}}{E(z)} \left(\cosh 2\frac{d_c}{\mathcal{R}_0} - 1\right)$$

(2c) Calling again n_G the galaxy density, the number of galaxies per square degree is easily integrated:

$$N = \int_0^5 n_G \frac{dV_c(z)}{dz} dz$$

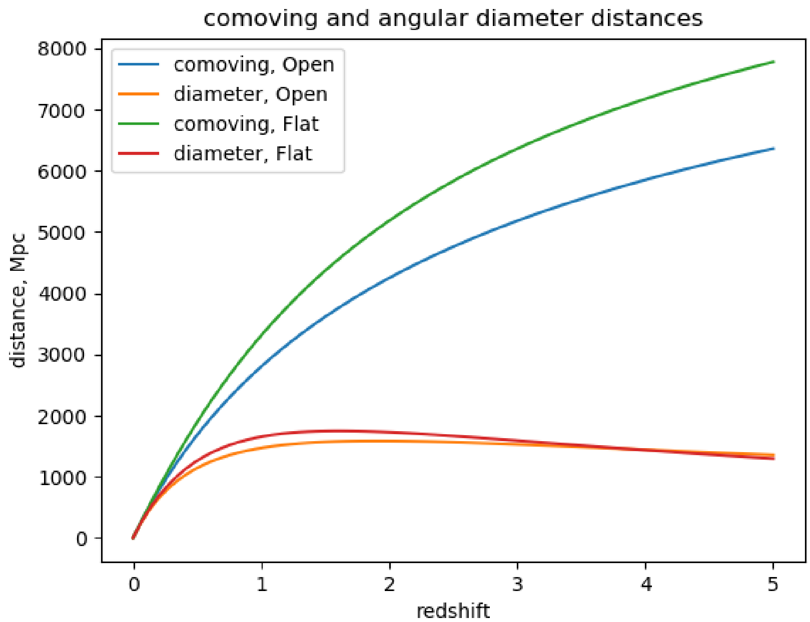

Figure 1:

The result is $N = 3.55 \times 10^5$ galaxies per square degree for the open model, $N = 4.78 \times 10^5$ for the flat model with cosmological constant.

(2d) The solid angle Ω_G covered by galaxies, with random orientations, is:

$$\Omega_G = \int_0^5 n_G \frac{dV_c(z)}{dz} 2 \left(\frac{R_G}{d_d(z)}\right)^2 dz$$

This results in $\Omega_G = 3.35 \times 10^{-5}$ and 4.11×10^{-5} steradians for the open and flat models, corresponding to fractions of 11.0% and 13.5% of one square degree. So galaxies in these models cover a fair fraction of the sky, something that was not apparent in the calculation before. One can try to estrapolate the result to even higher redshift, say to recombination ($z = 1100$) that defines the visible horizon; you will notice that the volume to the visible horizon is much larger than that to $z = 5$ (by a factor of 13 for open and 5 for flat cosmologies), so if one assumes that galaxies are present at recombination the number of galaxies will grow by the same factor, but the angle covered by galaxies will explode, going to values much larger than one. The reason why this happens is that we are placing galaxies of a 10 kpc proper radius on a surface at fixed distance that is smaller and smaller when we go to higher and higher redshift, and so at smaller and smaller scale factors. The reason why the Universe expansion solves Olber's paradox is that galaxies, and then stars, were born gradually in time, with a star-formation rate peaking at redshift $z \sim 2$; only $\sim 10\%$ of stars were born before $z = 5$.

(3a) The luminosity distance is reported in Figure 2, with the other curves. Luminosity distances are much larger than the other distances, so showing all of them together would have not worked if not using a log scale on the y axis. Notice how the luminosity distance at $z \sim 1$ is larger for the model with cosmological constant, see our discussion of the evidence of an accelerating universe.

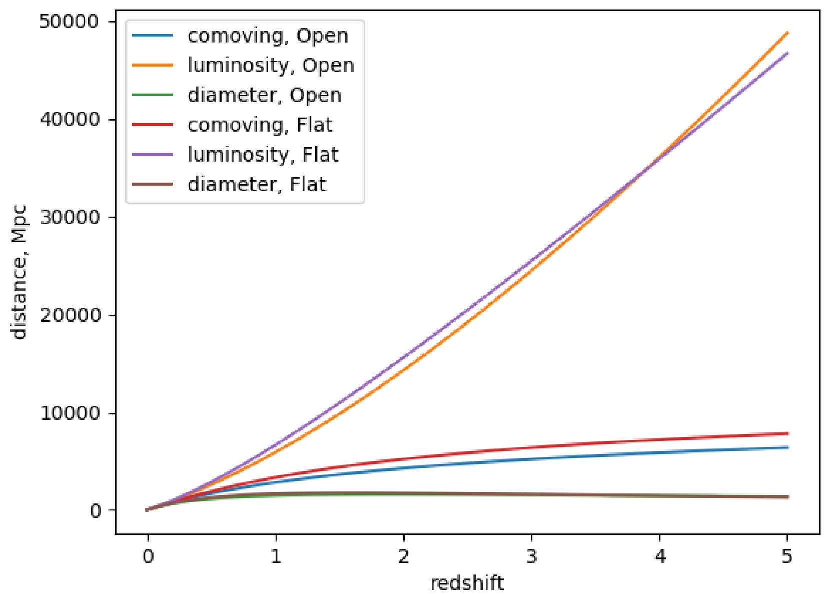

Figure 2:

(3b) The highest luminosity distance at which a galaxy of absolute magnitude $M_* = -20.5$ is visible is $d_l = 10.^{(m-M_*)/5-5}$ Mpc, the $z_{\rm max}(m)$ redshift can be found by numerically inverting the $d_L(z)$ function. The result is given in Figure 3 for the two fiducial models. At magnitude $m \simeq 26.6$ M_*, galaxies are visible down to redshift 5.

(3c) The number of galaxies is obtained by solving this integral:

$$N(m) = \int_0^{z_{\rm max}(m)} \Phi_*(m - 5\log_{10} d_l(z) - 25 - M_*) \frac{dV_c(z)}{dz} dz$$

The result is shown in Figure 4, while Figure 5 shows the same counts for a larger number of open and flat cosmologies. At $m < 20$ we are seeing only the local universe ($z < 1$), and number counts do not show differences for the two fiducial models. Conversely, deeper magnitudes allow us to probe the distant universe, and number counts starts to diverge, giving sizeable differences (Figure 5) that could in principle be used to constrain cosmological parameters.

(4) Number counts are very convenient from the point of view of observational time, because they just require an image of the sky in a single observational band. They do depend on cosmology, so they could in principle be used to constrain the cosmological parameters. However, several complications make this use impossible in practice. Galaxies do evolve in time, and their evolution is complicated and not understood in detail. Moreover, their evolution is different when galaxies are observed at different wavelengths. If their luminosity function were known in detail, it would be possible to compute their number density as a function of cosmology; but measuring the luminosity function requires a cosmological model to compute luminosities from fluxes. Another complication lies in the fact that the galaxy spectra are not flat at all, and redshift makes us see in a given observational band different "rest-frame" wavelengths, giving an

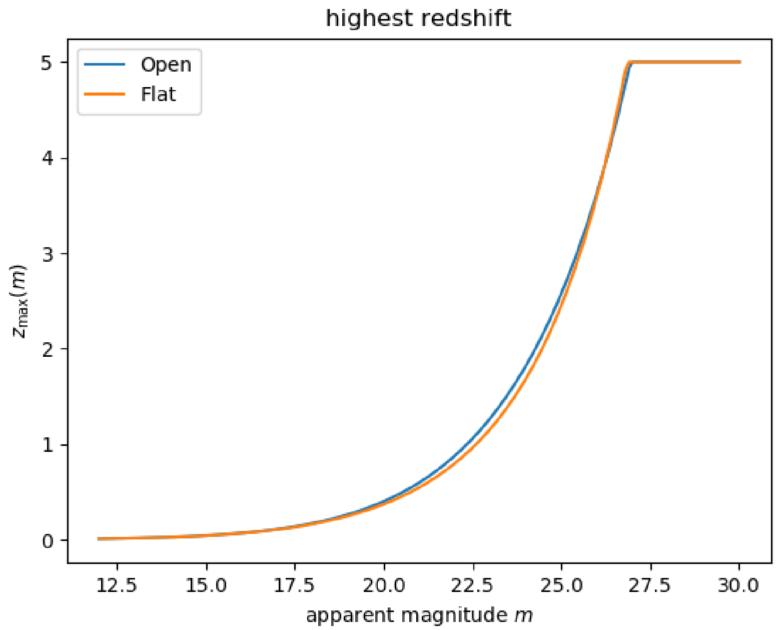

Figure 3:

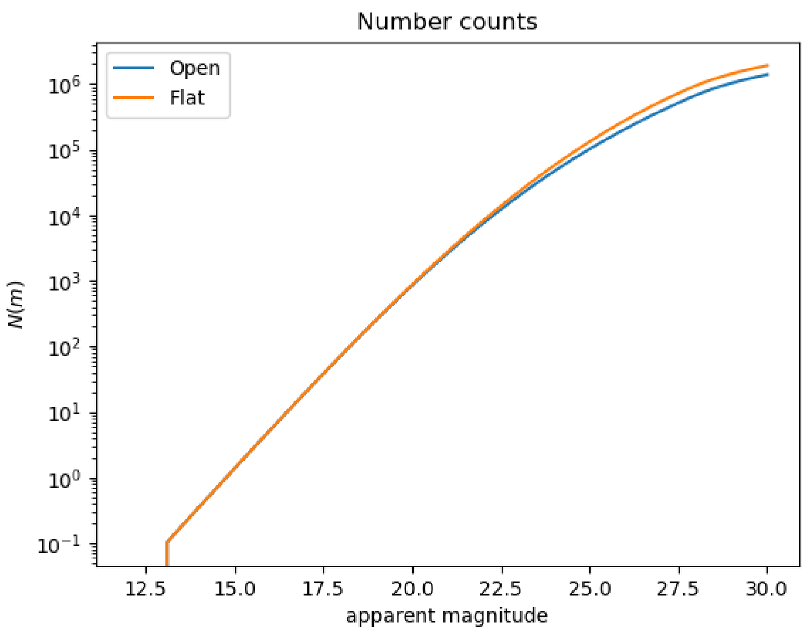

Figure 4:

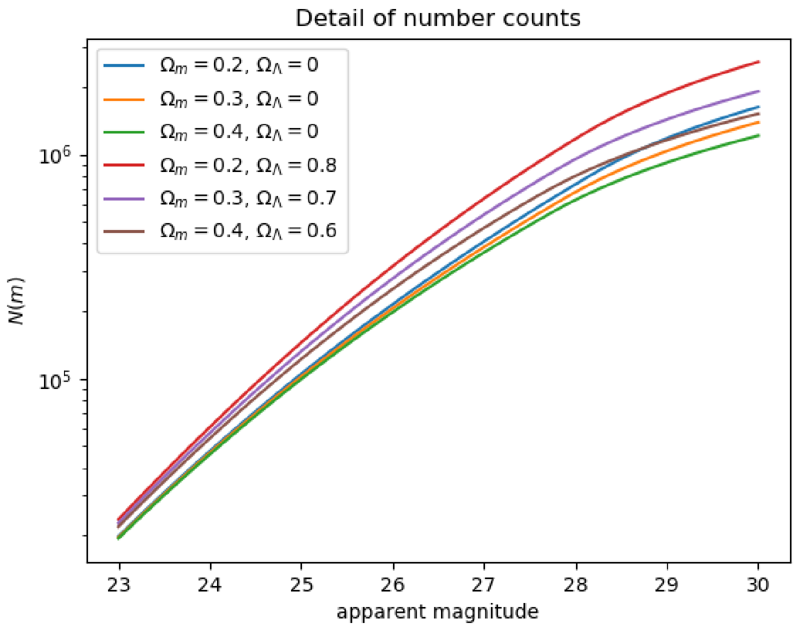

Figure 5:

apparent evolution of luminosities that depends on galaxy type.