Lecture 11 Flat and non-flat models

FLAT MODELS WITH A GENERIC EQUATION OF STATE

In cosmology it is customary to write the equation of state as:

$$P = w\rho \qquad (\text{or } P = w\rho c^2)$$

w is the EQUATION OF STATE PARAMETER

+ Matter, non relativistic: $P = nkT \ll \rho c^2 \Rightarrow w = 0$
+ Radiation, or relativistic matter: $P = \frac{1}{3}\rho c^2 \Rightarrow w = \frac{1}{3}$
+ Cosmological constant: $P = -\rho c^2 \Rightarrow w = -1$

NB: for a fluid, $v_s = \left(\frac{\partial P}{\partial \rho}\right)_s^{1/2} = \sqrt{w}\, c$ adiabatic sound speed

SOLUTION OF FRIEDMANN EQUATIONS:

(3) $\dot\rho = -3\frac{\dot a}{a}\left(\rho + \frac{P}{c^2}\right) = -3\frac{\dot a}{a}(1+w)\rho$

$$\frac{d\ln\rho}{dt} = -3(1+w)\frac{d\ln a}{dt} \Rightarrow \rho_w = \rho_{w0}\, a^{-3(1+w)}$$

Matter: $w = 0$, $\rho_m = \rho_{m0}\, a^{-3} = \rho_{m0}(1+z)^3$

Radiation: $w = \frac{1}{3}$, $\rho_r = \rho_{r0}\, a^{-4} = \rho_{r0}(1+z)^4$

(Note: ρ_r and ρ_{r0} are MATTER DENSITIES, $\rho_r = \frac{u_r}{c^2}$)

Interpretation: matter is subject to dilution
radiation to dilution + redshift

(2) $H^2 = \frac{8\pi G}{3}\rho_w$ (flat model)

$$\rho_w = \rho_c = \frac{3H^2}{8\pi G} = \rho_{w0}\, a^{-3(1+w)}$$

$$\Rightarrow H^2 = H_0^2\, a^{-3(1+w)}, \quad \frac{da}{dt} = H_0\, a^{-\frac{1+3w}{2}}$$

(Note: if $w = -\frac{1}{3}$, $\frac{da}{dt} = \text{const} = H_0$, $a = tH_0$,
age of the universe: $t_0 = \frac{1}{H_0}$)

Ansatz: $a(t) = \left(\frac{t}{t_0}\right)^{\alpha}$, $\frac{da}{dt} = \frac{\alpha}{t_0}\left(\frac{t}{t_0}\right)^{\alpha-1}$,

$$\frac{\alpha}{t_0}\left(\frac{t}{t_0}\right)^{\alpha-1} = H_0\left(\frac{t}{t_0}\right)^{-\alpha\left(\frac{1+3w}{2}\right)}$$

$$\Rightarrow \quad \alpha - 1 = -\alpha\left(\frac{1+3w}{2}\right) \quad \Rightarrow \quad \alpha = \frac{2}{3(1+w)} \qquad w \neq -1$$

Solution: $a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}$, $t_0 = \frac{2}{3(1+w)}\frac{1}{H_0}$

$$H(t) = \frac{2}{3(1+w)}\frac{1}{t} = \frac{H_0 t_0}{t}, \quad \rho_w(t) = \frac{1}{6(1+w)^2\pi G t^2}$$

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = \frac{1+3w}{2}$$

Distances:

$$L_{LOS} = \int_t^{t_0}\frac{c\,dt}{a(t)} = c\int_t^{t_0}\left(\frac{t}{t_0}\right)^{-\frac{2}{3(1+w)}}dt =$$

$$= c t_0\int_{t/t_0}^{1} x^{-\frac{2}{3(1+w)}}dx = 3ct_0\left(\frac{1+w}{1+3w}\right)\left[1-\left(\frac{t}{t_0}\right)^{\frac{1+3w}{3(1+w)}}\right]$$

$$= 3ct_0\left(\frac{1+w}{1+3w}\right)\left[1-(1+z)^{-\frac{1+3w}{2}}\right]$$

FLAT MODEL WITH RADIATION: $w = \frac{1}{3}$

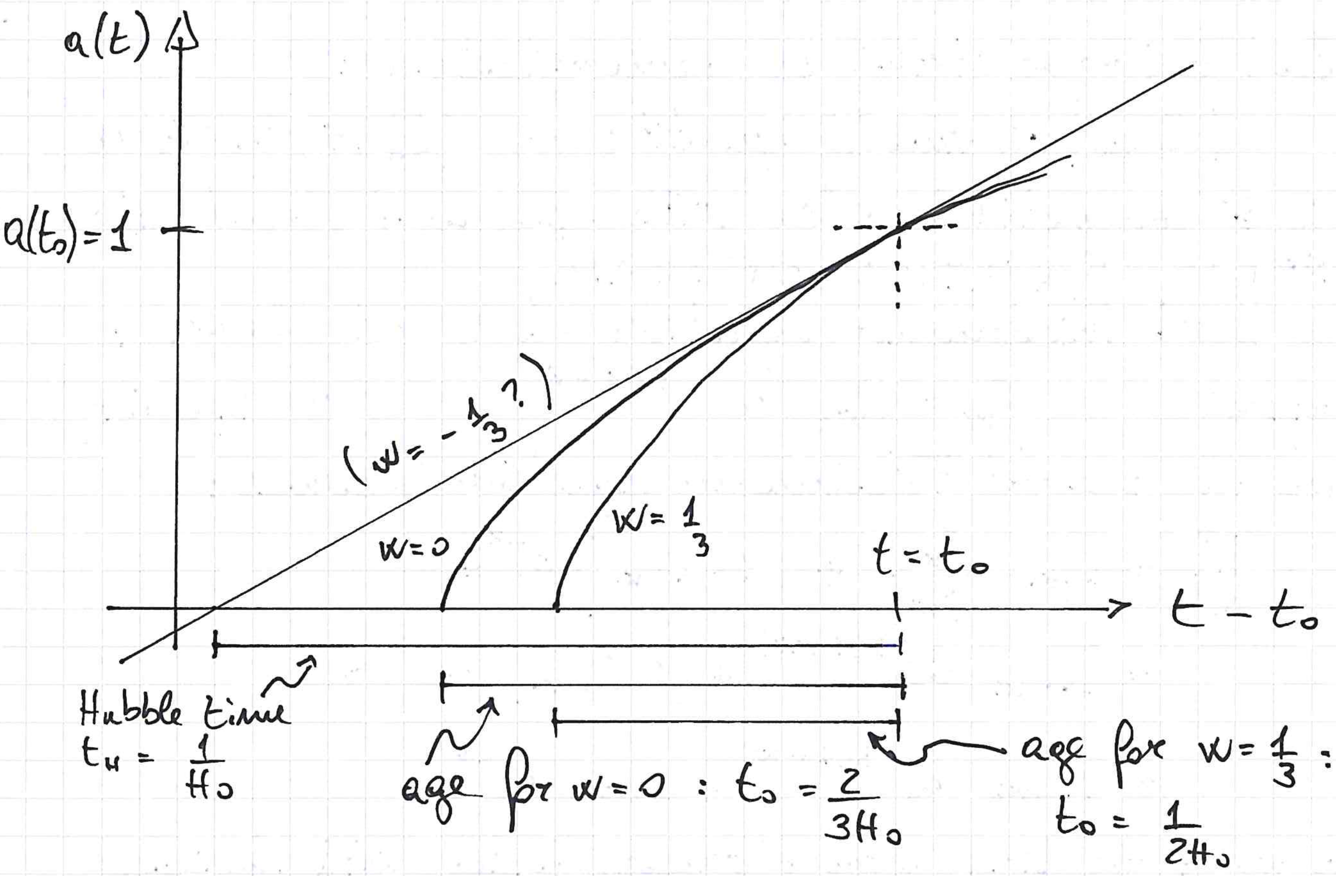
$$a(t) = \left(\frac{t}{t_0}\right)^{1/2}, \quad t_0 = \frac{1}{2H_0}, \quad H = \frac{1}{2t},$$

$$q_0 = 1, \quad \rho = \frac{3}{32\pi G t^2}$$

$$d_c = 2ct_0\left[1-\sqrt{\frac{t}{t_0}}\right] = 2ct_0\frac{z}{1+z}$$

$$d_L = 2ct_0 z, \quad d_D = 2ct_0\frac{z}{(1+z^2)}, \text{ max at } z=1$$

$$d_H = 2ct$$



CURVED MODELS

From the second Friedmann equation:

$$\Omega(t) \equiv \frac{\rho(t)}{\rho_c(t)} = 1 + k \frac{c^2}{R_0^2 a^2 H^2}$$

Let's define:

$$\Omega_k = 1 - \Omega \quad \text{such that} \quad \Omega + \Omega_k = 1$$

NB Ω contains ALL forms of energy here

$$\Omega_k = -k \left(\frac{c}{H}\right)^2 \frac{1}{(aR_0)^2} =$$

$$= -k \times \left(\frac{\text{horizon size}}{\text{universe curvature radius}} \right)^2$$

$$\Omega_{k0} = 1 - \Omega_0 = -\frac{kc^2}{R_0^2 H_0^2} \Rightarrow R_0 = \frac{c}{H_0} \frac{1}{\sqrt{|\Omega_{k0}|}}$$

Closed universe: $k = 1, \ \Omega_0 > 1, \ \Omega_{k0} < 1$

Open universe: $k = -1, \ \Omega_0 < 1, \ \Omega_{k0} > 1$

Flat universe: $R_0 \to \infty$

A flat universe cannot be demonstrated:

$$|\Omega_{k0}| < \text{small quantity} \Rightarrow R_0 \gg \text{horizon}$$

this is compatible also with open and closed universes

GENERIC EQUATION OF STATE

The third Friedmann equation does not change:

suppose $\rho_w = \rho_{w0}\, a^{-3(1+w)}$ (one species)

Then the second Friedmann equation becomes:

$$\frac{H^2}{H_0^2} = \Omega_{w0} \frac{\rho_w}{\rho_{w0}} + (1-\Omega_{w0})\, a^{-2} =$$

$$= \Omega_{w0}\, a^{-3(1+w)} + (1-\Omega_{w0})\, a^{-2} =$$

$$= \Omega_{w0}\, a^{-3(1+w)} + \Omega_{k0}\, a^{-2}$$

For $w > -\frac{1}{3}$ the first term decreases faster with the scale factor than the second curvature term.
So the two are equal when:

$$\Omega_{w0}\, a^{-3(1+w)} = |\Omega_{w0} - 1|\, a^{-2}$$

$$\Rightarrow \quad a^* = \left|\frac{1}{\Omega_{w0}} - 1\right|^{-\frac{1}{1+3w}}$$

$a \ll a^*$: curvature is negligible, $a \propto t^{\frac{2}{3(1+w)}}$

$a \gg a^*$: curvature is dominant

MILNE MODEL: empty universe, $P = \rho = 0$

$$H^2 = H_0^2\, a^{-2} \Rightarrow \frac{da}{dt} = H_0 \Rightarrow a = H_0 t, \quad t_0 = \frac{1}{H_0}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) = 0 \quad \Rightarrow \quad \ddot{a} = 0, \quad q = 0$$

Curvature is the only contributor to Friedmann equations

An empty universe MUST be open!

Note: $w = -\frac{1}{3}$ mimics an empty universe

CLOSED UNIVERSE, MATTER-DOMINATED:

$\Omega_0 > 1$, $\Omega_{k0} < 1$, $a = a^*$: $\frac{da}{dt} = 0$

$\Rightarrow$ the scale factor has a maximum

$$\frac{H^2}{H_0^2} = \Omega_0 a^{-3} + (1-\Omega_0) a^{-2}$$

$$\Rightarrow \left(\frac{da}{dt}\right)^2 = H_0^2 \left(\frac{\Omega_0}{a} + 1 - \Omega_0\right)$$

This is analogous to the cycloid equation:

$$\left(\frac{dy}{dx}\right)^2 = \frac{2r}{y} - 1 \qquad (r \text{ is a parameter})$$

that admits a parametric solution:

$$x = r(\vartheta - \sin\vartheta)$$
$$y = r(1 - \cos\vartheta)$$

It is easy to recast Friedmann equation to the form of the cycloid, to obtain:

$$a(\vartheta) = \frac{\Omega_0}{2(\Omega_0 - 1)} (1 - \cos\vartheta) \qquad 0 \leq \vartheta \leq 2\pi$$

$$t(\vartheta) = \frac{1}{2H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} (\vartheta - \sin\vartheta)$$

$a(t)$ grows from $\vartheta = 0$ to $\vartheta = \pi$, where it has a maximum at a time:

$$t_m = \frac{\pi}{2H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} , \quad a_m = \frac{\Omega_0}{\Omega_0 - 1} = a^*$$

The age of the universe can be obtained by computing the ϑ_0 at which $a = 1$:

$$\frac{\Omega_0}{2(\Omega_0 - 1)} (1 - \cos\vartheta_0) = 1 \quad \Rightarrow \quad \cos\vartheta_0 = \frac{2 - \Omega_0}{\Omega_0}$$

$$\sin\vartheta_0 = \frac{2}{\Omega_0}\sqrt{\Omega_0 - 1}$$

$$\Rightarrow t_0 = \frac{1}{2H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} \left[\operatorname{acos}\left(\frac{2 - \Omega_0}{\Omega_0}\right) - \frac{2}{\Omega_0}\sqrt{\Omega_0 - 1} \right]$$

One can compute that: $t_0 \xrightarrow[\Omega_0 \to 1]{} \frac{2}{3H_0}$

and $t_0 < \frac{2}{3H_0}$

At $t = 2t_m$: $a(t) = 0$ BIG CRUNCH

OPEN UNIVERSE, MATTER DOMINATED

$$\Omega_0 < 1, \quad \Omega_{k0} > 1, \quad a = a^* : \quad \frac{da}{dt} > 0$$

Using the same technique as for the closed model:

$$a(\vartheta) = \frac{\Omega_0}{2(1-\Omega_0)}(\cosh\vartheta - 1)$$

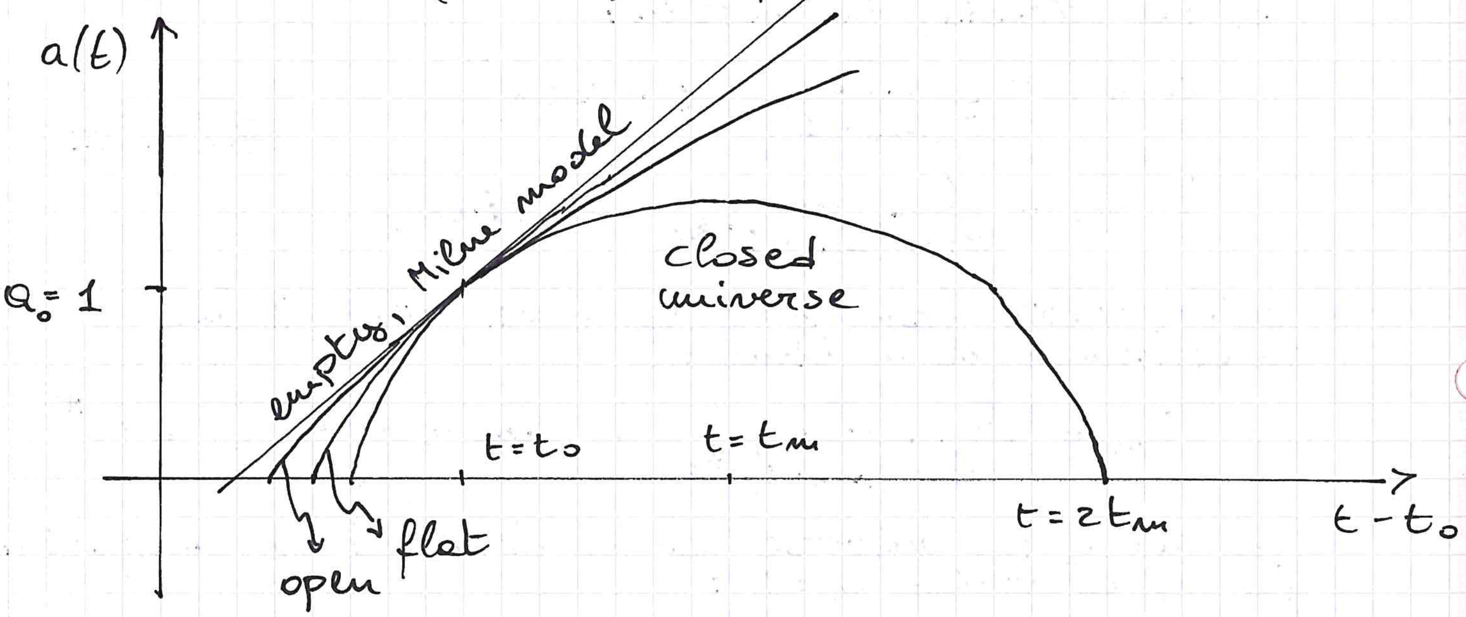
$$t(\vartheta) = \frac{1}{2H_0}\frac{\Omega_0}{(1-\Omega_0)^{3/2}}(\sinh\vartheta - \vartheta)$$

$$t_0 = \frac{1}{2H_0}\frac{\Omega_0}{(1-\Omega_0)^{3/2}}\left[\frac{2}{\Omega_0}\sqrt{1-\Omega_0} - \operatorname{acosh}\left(\frac{2-\Omega_0}{\Omega_0}\right)\right]$$

$$t_0 \xrightarrow[\Omega_0 \to 1]{} \frac{2}{3H_0}, \quad t_0 \xrightarrow[\Omega_0 \to 0]{} \frac{1}{H_0}, \quad t_0 > \frac{2}{3H_0}$$

At $a \gg a^*$:

$$H^2 \simeq H_0^2(1-\Omega_0)a^{-2}, \quad a = H_0\sqrt{1-\Omega_0}\, t$$



An open model has $\frac{2}{3H_0} < t_0 < \frac{1}{H_0}$, so it can solve, or make less severe, an age problem

DECELERATION PARAMETER

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho = -\frac{H_0^2}{2}\frac{\rho_0}{\rho_{c0}}a^{-3} = -\frac{1}{2}\Omega_0 H_0^2 a^{-3}$$

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = \frac{\frac{1}{2}\Omega_0 H_0^2 a^{-3}}{H_0^2(\Omega_0 a^{-3} + (1-\Omega_0)a^{-2})} = \frac{1}{2}\frac{\Omega_0/a}{\Omega_0/a + 1 - \Omega_0}$$

Let's now compute the evolution of the density parameter $\Omega(t)$:

$$\Omega(a) = \frac{8\pi G}{3H^2}\rho = \frac{8\pi G}{3} \frac{\rho_0 a^{-3}}{H_0^2(\Omega_0 a^{-3} + (1-\Omega_0)a^{-2})} =$$

$$= \frac{\Omega_0/a}{\Omega_0/a + (1-\Omega_0)}$$

$$\Rightarrow \quad q = \frac{1}{2}\Omega$$

The evolution of Ω can also be written as:

$$\frac{1}{\Omega} - 1 = a\left(\frac{1}{\Omega_0} - 1\right)$$

Plotted in $\log t$: