Lecture 12 Models with Λ

FRIEDMANN EQUATIONS WITH Λ

When Λ is not hidden in $T^{\alpha\beta}$:

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}, \quad \text{trace: } R = -\frac{8\pi G T}{c^4} + 4\Lambda$$

$$\Rightarrow \quad R_{\alpha\beta} = \frac{8\pi G}{c^4}\left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}\right) + \Lambda g_{\alpha\beta}$$

00 component, adding c:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3\frac{P}{c^2}\right) + \frac{\Lambda c^2}{3}$$

ij components, adding c:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}$$

This is consistent with a fluid with:

$$\rho_\Lambda = \frac{\Lambda c^2}{8\pi G}, \quad P_\Lambda = -\frac{\Lambda c^4}{8\pi G} = -\rho_\Lambda, \quad w = -1$$

The third Friedmann equation does not change:
+ either Λ is out of $T^{\alpha\beta}$
+ or if is a constant and does not contribute to $T^{\alpha\beta}{}_{;\alpha}$

EFFECT OF Λ: $\quad \frac{\ddot{a}}{a} > 0$ if $\frac{\Lambda c^2}{3} > \frac{4\pi G}{3}\left(\rho + 3\frac{P}{c^2}\right)$

NB: for $P = w\rho c^2$, $\quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho\,(1+3w)$

$$\Rightarrow \quad \frac{\ddot{a}}{a} > 0 \text{ if } w < -\frac{1}{3}, \text{ surely for } w = -1$$

EINSTEIN STATIC UNIVERSE

To have a static universe we need $\dot{a} = 0$, $\ddot{a} = 0$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho + \frac{\Lambda c^2}{3} = 0 \quad \Rightarrow \quad \frac{\Lambda c^2}{3} = \frac{4\pi G}{3}\rho$$

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2 R_E^2} + \frac{\Lambda c^2}{3} = 0 \quad \Rightarrow \quad \frac{\Lambda c^2}{3} = \frac{kc^2}{a^2 R_E^2} - \frac{8\pi G}{3}\rho$$

This can be true if $k=1$ (closed universe):

$$R_E^2 = \frac{c^2}{4\pi G\rho}, \quad \Lambda = \frac{1}{R_E^2}$$

This model thus requires tuning of ρ, Λ and R_E.
But $\rho \propto a^{-3}$, so if a grows then $\frac{\Lambda c^2}{3} > -\frac{4\pi G}{3}\rho$
$\Rightarrow \ddot{a} > 0$
This solution is thus unstable.

DE SITTER MODEL

Consider a flat universe with only cosmological constant:

$$H^2 = \frac{\dot{a}^2}{a^2} = \frac{\Lambda c^2}{3} \Rightarrow \frac{da}{dt} = \sqrt{\frac{\Lambda c^2}{3}}\, a \,,\; a(t) = e^{\sqrt{\frac{\Lambda c^2}{3}}(t-t_0)}$$

This is an exponential growth with constant H.
But what is t_0 here?

$$a(t=0) = e^{-Ht_0} \neq 0$$

$a \xrightarrow[t\to-\infty]{} 0$ the Big Bang is moved to $-\infty$, so there is no BB here

Comoving particle horizon:

$$d_H = \int_{-\infty}^{t_0} \frac{c\,dt}{a(t)} = \frac{c}{H}\int_{-\infty}^{0} e^{-x}\,dx \to \infty$$

conversely, photons emitted today cannot travel an infinite comoving distance, even in an infinite time:

$$d_{EH} = \int_{t_0}^{\infty} \frac{c\,dt}{a(t)} = \frac{c}{H}\int_0^{\infty} e^{-x}\,dx = \frac{c}{H}$$

so a de Sitter model has an EVENT HORIZON: one cannot communicate beyond $\frac{c}{H}$

MODELS WITH MATTER, RADIATION, CURVATURE, Λ

We know that the universe contains:

- matter (baryons + dark matter + neutrinos)
- radiation (photons + neutrinos at high redshift + ultra-relativistic matter at very early times)
- dark energy as a cosmological constant
- curvature, consistent with zero

The second Friedmann equation is written as:

$$E^2 = \frac{H^2}{H_0^2} = \Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_k a^{-2} + \Omega_\Lambda$$
$$\Omega_m + \Omega_r + \Omega_k + \Omega_\Lambda = 1$$

NB: should be Ω_{m0}, Ω_{r0} etc., I dropped the "0" pedix to have a lighter notation

Measurements give $\Omega_r \ll 1$, $\Omega_k \simeq 0$

LAMDA CDM MODEL ("CONCORDANCE" MODEL):

$$E^2 = \frac{H^2}{H_0^2} = \Omega_m a^{-3} + \Omega_\Lambda \qquad \Omega_m = 0.31$$
$$\Omega_\Lambda = 0.69$$

NB: $\rho_\Lambda = \frac{\Lambda c^2}{8\pi G}$ is a constant

$\Omega_\Lambda(z) = \frac{\rho_\Lambda}{\rho_c}$ is not a constant because ρ_c is a function of time (or z)

$= \frac{\Lambda c^2}{3H^2}$

but if $\rho_\Lambda = \rho_c \Rightarrow H^2 = \frac{\Lambda c^2}{3} = \text{const}$

as in the De Sitter model, $\Omega_\Lambda = 1$

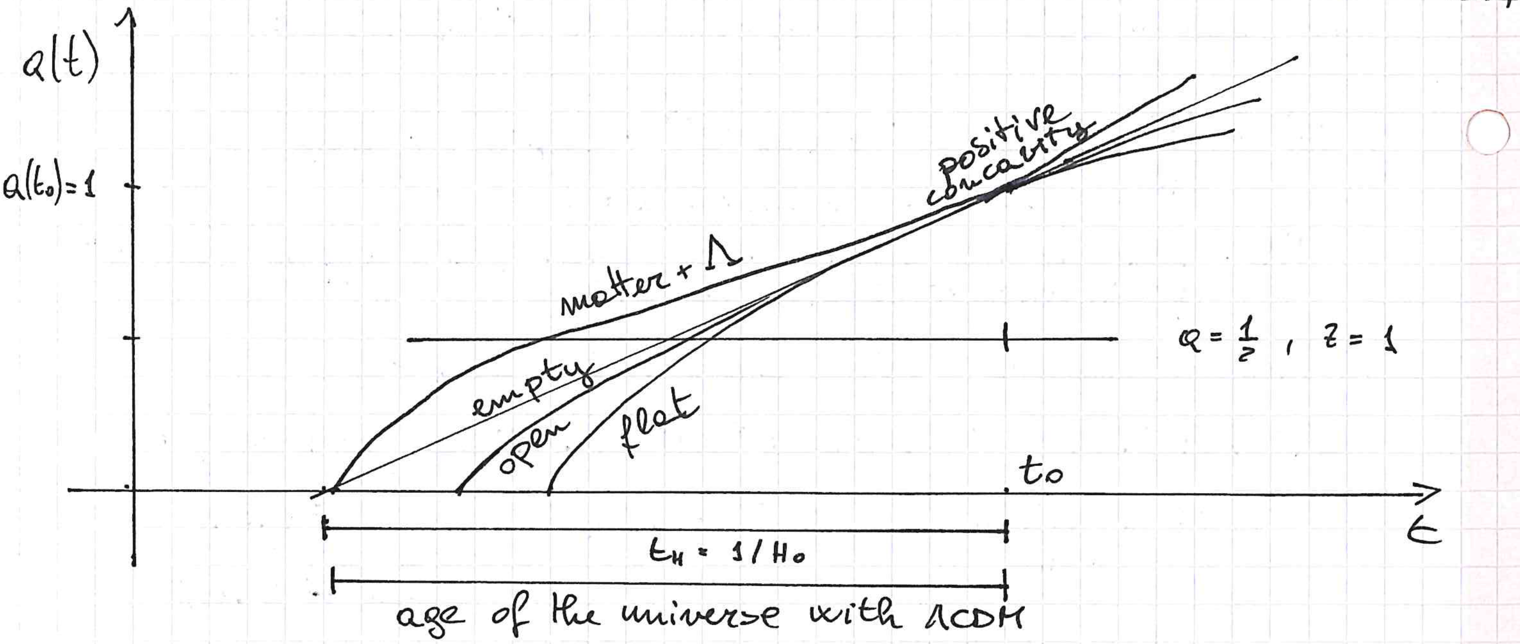
It is possible to find a parametric solution to this equation, that leads to an analytic solution:

$$a(t) = \left(\frac{1}{\Omega_m} - 1\right)^{-1/3} \sinh^{2/3}\left(\frac{3}{2} t H_0 \sqrt{1 - \Omega_m}\right)$$

The resulting age of the universe is:

$$t_0 = \frac{2}{3H_0\sqrt{1-\Omega_m}} \sinh^{-1}\left(\sqrt{\frac{1}{\Omega_m} - 1}\right)$$

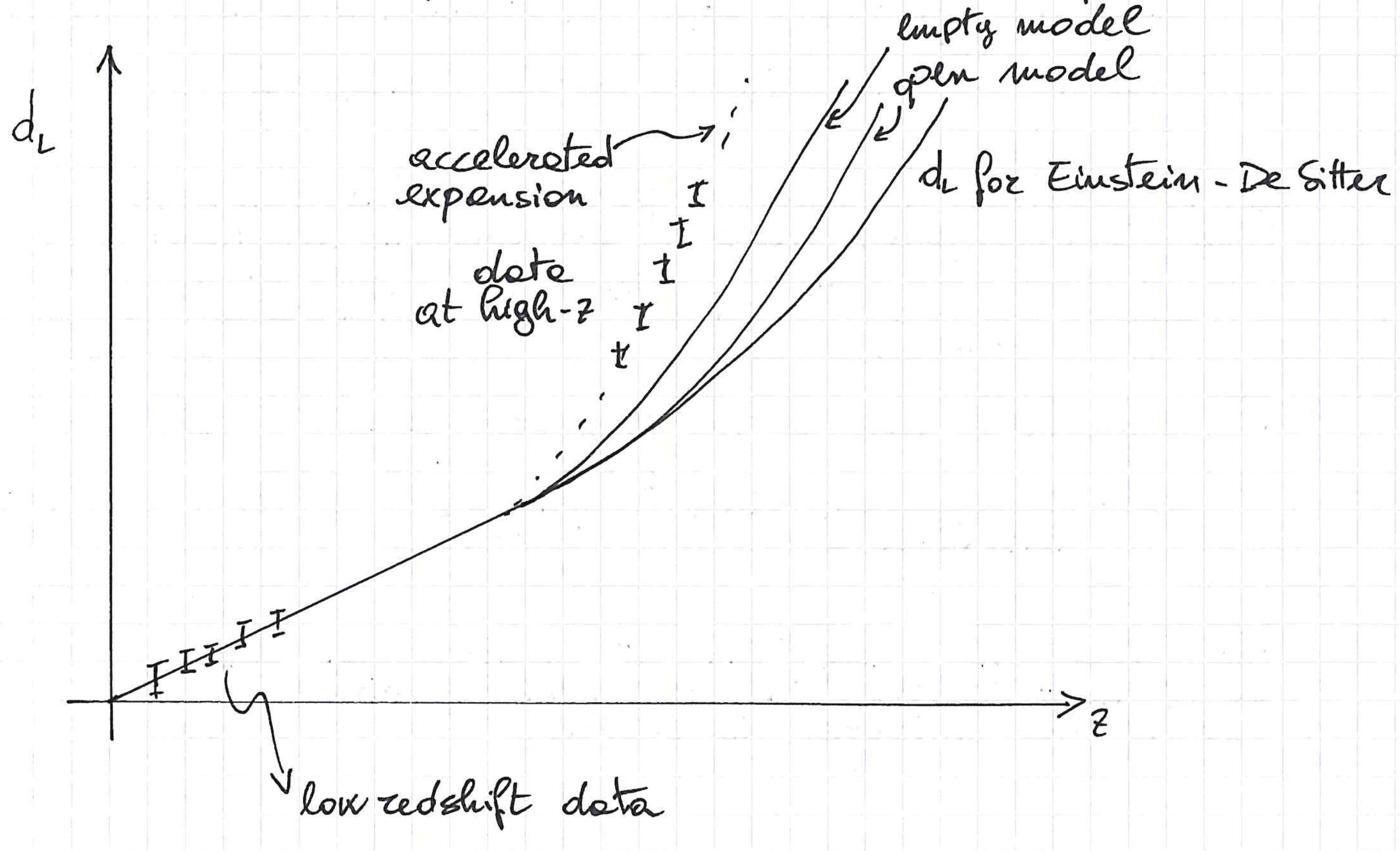
For $\Omega_m \simeq 0.3$, $\Omega_\Lambda \simeq 0.7$ we have $t_0 \simeq t_H$, so if $H_0 \simeq 70$ km/s/Mpc the problem with the age of the Universe is solved



If the universe is accelerating, a standard candle at $z=1$ would shine earlier than even an empty model.

As a consequence the luminosity distance in ΛCDM is larger than that of an empty model.

The Hubble diagram of type Ia SNe gives an observational demonstration of the Universe accelerated expansion.



The first Friedmann equation can be written as:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho_m + \frac{\Lambda c^2}{3} = -\frac{1}{2}\Omega_m H_0^2 a^{-3} + H_0^2 \Omega_\Lambda$$

so the q parameter is:

$$q = -\frac{\ddot{a}}{a}\frac{a^2}{\dot{a}^2} = \frac{1}{2}\Omega_m \frac{H_0^2}{H^2} a^{-3} - \frac{H_0^2}{H^2}\Omega_\Lambda$$

for $t = t_0$, $q_0 = \frac{1}{2}\Omega_m - \Omega_\Lambda$, negative for $\Omega_\Lambda > \frac{1}{2}\Omega_m$

Matter - dark energy equivalence:

$$\rho_\Lambda = \rho_m \quad , \quad \frac{\rho_\Lambda}{\rho_{m0} a^{-3}} = 1 = \left(\frac{\Omega_\Lambda}{\Omega_m}\right)(1+z)^{-3}$$

$$1 + z_{de,eq} = \left(\frac{\Omega_\Lambda}{\Omega_m}\right)^{1/3} \simeq 1.31 \quad , \quad z_{de,eq} \simeq 0.31$$

Start of acceleration:

$$\frac{\Omega_\Lambda}{\Omega_{m}(z)} = \frac{1}{2} = \left(\frac{\Omega_\Lambda}{\Omega_m}\right)(1+z)^{-3} \quad , \quad 1 + z_{acc} = \left(\frac{2\Omega_\Lambda}{\Omega_m}\right)^{1/3} \simeq 1.65$$

$$z_{acc} \simeq 0.65$$

NON-FLAT ΛCDM MODELS:

For $\Lambda = 0$ we have that

open model → expand forever

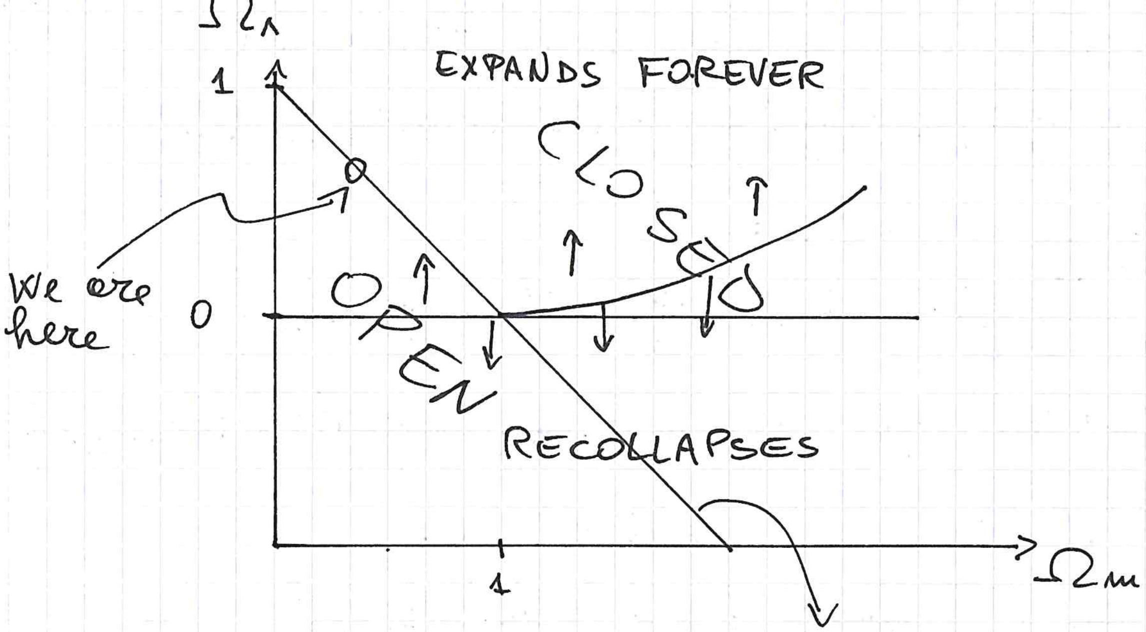
closed model → recollapses

With $\Lambda \neq 0$ this is not true any more. To see what universes can recollapse we can compute if $a(t)$ has a maximum at a^*:

$$\frac{da}{dt} = 0 \Rightarrow H = 0 \Rightarrow \Omega_m a^{*-3} + \Omega_\Lambda + (1 - \Omega_m - \Omega_\Lambda) a^{*-2} = 0$$

This cubic equation has a solution if:

$$\Omega_\Lambda \geq \begin{cases} 0 & 0 \leq \Omega_m \leq 1 \\ 4\Omega_m \cos^3\left[\frac{1}{3}\operatorname{acos}\left(\frac{1-\Omega_m}{\Omega_m}\right) + \frac{4\pi}{3}\right] & \Omega_m > 1 \end{cases}$$



$\Omega_m + \Omega_\Lambda = 1$, flat models