Lecture 15 Thermodynamics of the early Universe

Bonometto, Chapter IV

Statistics of relativistic particles

Let's call dN the number of particles that at a time t are found in a phase-space volume around $(\vec{x}, \vec{p})$
NB: here $\vec{x}$ and $\vec{p}$ are three-vectors

$$dN = f(\vec{x}, \vec{p}, t)\, d^3x\, d^3p$$

In a FRW f cannot depend on $\vec{x}$ (homogeneity) and on the direction of $\vec{p}$, it can depend only on $p \equiv |\vec{p}|$

NB: only in this discussion p is the momentum module, not the pressure

From statistical mechanics:

$$f(p) = \frac{N_s}{(2\pi\hbar)^3} \frac{1}{\exp\frac{E(p)}{k_B T} \pm 1} \qquad E^2(p) = p^2c^2 + m^2c^4$$

+ : fermions
− : bosons

The energy density is:

$$u = \int d^3p\, E(p) f(p) = \frac{N_s 4\pi}{(2\pi\hbar)^3} \int_0^\infty dp\, p^2 \frac{E(p)}{\exp\frac{E(p)}{k_B T} \pm 1}$$

Let's consider the case $pc \gg mc^2$, that applies to ultra-relativistic matter and to radiation:

$$u = \frac{N_s 4\pi}{(2\pi)^3 \hbar^3} \int_0^\infty dp\, p^2 \frac{pc}{\exp\left(\frac{pc}{k_B T}\right) \pm 1} =$$

$$= \frac{N_s}{2\pi^2} \frac{(k_B T)^4}{(\hbar c)^3} \int_0^\infty \frac{x^3 dx}{e^x \pm 1} = \frac{\pi^2}{30} \begin{pmatrix} N_s \\ \frac{7}{8} N_s \end{pmatrix} \frac{(k_B T)^4}{(\hbar c)^3}$$

Here N_s is the number of allowed spin states (or polarizations) and the two options in parenthesis refer to bosons (N_s) or fermions $\left(\frac{7}{8} N_s\right)$. A similar integral must be solved for pressure, obtaining:

$$P = \frac{1}{3} \int d^3p\, pc\, f(p)$$

$$\hookrightarrow P = \frac{1}{3} u$$

The particle number density results:

$$n = \frac{N_s}{(2\pi\hbar)^3} 4\pi \int_0^\infty dp\, p^2 \frac{1}{\exp \frac{pc}{k_B T} \mp 1} = \frac{\zeta(3)}{\pi^2} \begin{pmatrix} N_s \\ \frac{3}{4} N_s \end{pmatrix} \left(\frac{k_B T}{\hbar c}\right)^3$$

Here $\zeta(3)$ is Riemann's zeta function and is $\zeta(3) \simeq 1.202\ldots$

We can define the radiation constant $a_r = \frac{\pi^2 k_B^4}{15 \hbar^3 c^3}$

(usually called a, here we call it a_r not to confuse it with the scale factor)

$$\Rightarrow \quad u = \frac{1}{2} \begin{pmatrix} N_s \\ \frac{7}{8} N_s \end{pmatrix} a_r T^4 \;, \quad n = \frac{15\, \zeta(3)}{\pi^4} \begin{pmatrix} N_s \\ \frac{3}{4} N_s \end{pmatrix} \frac{a_r T^3}{k_B}$$

Electromagnetic radiation:

In this case $m=0$, so these relations always apply at equilibrium. For a boson with $N_s = 2$:

$$u = a_r T^4 \;, \quad n = \frac{30\, \zeta(3)}{\pi^4} \frac{a_r T^3}{k_B} \;, \quad P = \frac{1}{3} a_r T^4$$

For the CMB at $z=0$: $T = 2.73\ \mathrm{K}$

$$n_{\gamma 0} = 412\ \mathrm{cm}^{-3} \;, \quad u_{\gamma 0} = 4.2 \times 10^{-13}\ \mathrm{erg\ cm}^{-3}$$

$$\rho_{\gamma 0} = \frac{u_{\gamma 0}}{c^2} = 4.67 \times 10^{-34}\ \mathrm{g\ cm}^{-3}$$

Thermal soup:

In the early Universe we have several species of relativistic particles in equilibrium, that contribute to the energy density.

We write this as:

$$u = \frac{\pi^2}{30} g^* \frac{(k_B T)^4}{(\hbar c)^3} \;, \quad g^* = N_{bos} + \frac{7}{8} N_{fer}$$

so g^* gives the sum of all possible states of bosons $+ \frac{7}{8}$ of all the possible state of fermions

$$n = \frac{\zeta(3)}{\pi^2} \tilde{g} \left(\frac{k_B T}{\hbar c}\right)^3 \;, \quad \tilde{g} = N_{bos} + \frac{3}{4} N_{fer}$$

To these we must add:
- \+ non-relativistic particles
- \+ decoupled particles
- \+ particles that never were in equilibrium

Thermodynamics of an expanding volume

We know from basic thermodynamics that:

$$dU = -p\,dV + T\,dS$$

Here p is pressure again. This relation means: the internal energy U varies like $-PdV$ for a transformation at constant entropy S, like TdS at constant V

We apply this relation to an expanding, "comoving" volume.

Lets call $u \equiv \frac{dU}{dV}$ and $\sigma \equiv \frac{dS}{dV}$ the energy and entropy densities. We can write:

$$\sigma = \frac{1}{T}\,\frac{dU + p\,dV}{dV} = \frac{u+P}{T}$$

It is possible to obtain a relation between σ and P as follows.

Consider the free energy: $F = U - TS$

$$dF = dU - SdT - TdS = -p\,dV - S\,dT$$

$$\Rightarrow \frac{\partial F}{\partial V} = -P, \quad \frac{\partial F}{\partial T} = -S$$

$$\frac{\partial}{\partial T}\frac{\partial F}{\partial V} = -\frac{\partial P}{\partial T} = \frac{\partial}{\partial V}\frac{\partial F}{\partial T} = -\frac{dS}{dV} = -\sigma$$

$$\Rightarrow \sigma = \frac{\partial P}{\partial T}, \quad u = -P + T\frac{\partial P}{\partial T}$$

For a relativistic fluid: $P \propto T^4$, $\frac{\partial P}{\partial T} = 4\frac{P}{T}$

$$\Rightarrow u = 3P, \quad \sigma = \frac{4}{3}\frac{u}{T}$$

We can then write:

$$\sigma = \frac{4\pi^2}{90}\begin{pmatrix} N_s \\ \frac{7}{8}N_s \end{pmatrix} k_B \left(\frac{k_B T}{\hbar c}\right)^3 = \frac{4\pi^4}{90\,\zeta(3)}\begin{pmatrix} 1 \\ \frac{7}{6} \end{pmatrix} k_B n$$

For bosons: $\sigma = \frac{4}{3} a_r T^3 \simeq 3.60\, k_B n$

For a thermal soup: $\frac{\sigma}{k_B n} = \frac{4\pi^4}{90\,\zeta(3)}\,\frac{g^*}{\tilde{g}}$

For constant g^* and $\tilde{g}$: $\sigma \propto u$, $u \propto a^{-3}$

$\Rightarrow$ $S = a^3\sigma$ is conserved

Let's define: $\eta = \frac{n_b}{n_\gamma} = 3.60\, k_B \frac{n_b}{\sigma_\gamma} \simeq 2.68\times10^{-8}\Omega_b h^2$
$\simeq 5.97\times10^{-10}$

When both n_b and n_γ vary $\propto a^{-3}$, η is conserved.

$\eta^{-1} \propto \frac{\sigma_\gamma}{n_b}$ is proportional to the entropy per baryon.

(photons dominate the entropy at late times)

Entropy conservation

If S is conserved, then $dS = 0$ and we get back to

$$dU = -p\,dV$$

consistent with third Friedmann equation:

$$\dot{\rho}c^2 + 3\frac{\dot{a}}{a}(\rho c^2 + p) = 0$$

This can be developed as:

$$\frac{d}{dt}(\rho c^2 a^3) + p\frac{d}{dt}(a^3) = 0 \Rightarrow \frac{d}{dt}\left[(\rho c^2 + p)a^3\right] = a^3\frac{dp}{dt}$$

Now: $S = \sigma a^3 = \frac{u+p}{T}a^3 = \frac{\rho c^2 + p}{T}a^3$

$$\dot{S} = \frac{d}{dt}\left[\frac{\rho c^2 + p}{T}a^3\right] = \frac{a^3}{T}\dot{p} - S\frac{\dot{T}}{T}$$

Suppose $p = p(T)$, $\dot{p} = \frac{\partial p}{\partial t}\dot{T}$, $\sigma = \frac{\partial p}{\partial T}$

$$\Rightarrow \frac{a^3}{T}\dot{p} = a^3\sigma\frac{\dot{T}}{T} = S\frac{\dot{T}}{T} \Rightarrow \frac{dS}{dT} = 0$$

So if p is only a function of T and decreases because T decreases, the entropy of a comoving volume $S = a^3\sigma$ is conserved

Thermal soup

$$\sigma = g^*(T)\,\frac{2}{3}\,a_r T^3$$

where it has been made explicit that g^* is a function of T, because particles can annihilate or decouple when T decreases.

So the conservation of $S = \sigma a^3$ implies that:

$$g^{*\,1/3}\, a T = \text{const}$$

for most of the Universe history

Temperature jump at annihilation

Suppose that g^* varies quickly (with respect to expansion, that is on a timescale $\ll H^{-1}$) for instance because some particle-antiparticle pairs annihilate when $k_B T \sim mc^2$

Then, calling (−) the time before and (+) the time after the annihilation:

$$S_{(-)} = S_{(+)} \Rightarrow g^{*\,1/3}_{(-)}\, T_{(-)} = g^{*\,1/3}_{(+)}\, T_{(+)}$$

The decrease of g^* due to annihilation gives a temperature jump:

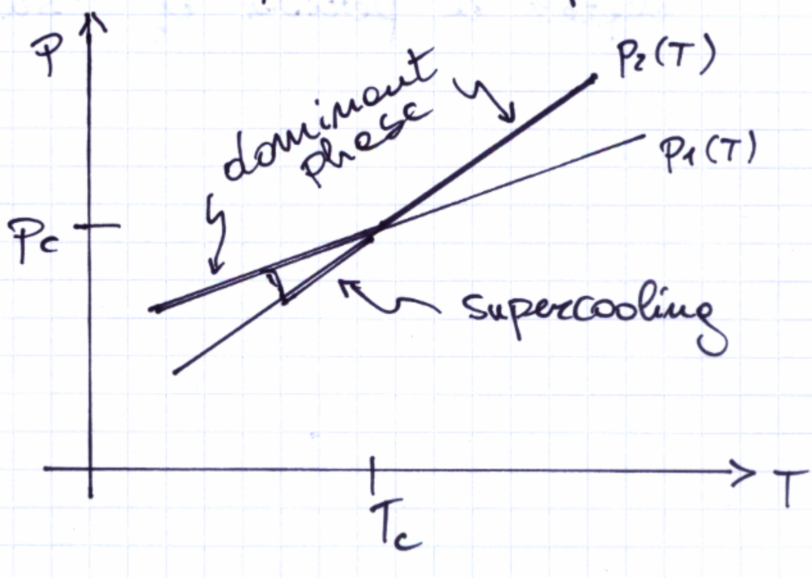
$$T_{(+)} = \left(\frac{g^*_{(-)}}{g^*_{(+)}} \right)^{1/3} T_{(-)}$$

This relation will be crucial to understand the temperature of the neutrino background.

Entropy production during a phase transition

Suppose that two phases are present in a fluid, with pressures

$p_1(T)$, $p_2(T)$



The dominant phase is that at higher pressure, because it can enucleate bubbles in the other phase

Phase transition can take place at

$T = T_c$, $P = P_c$

where $p_1(T_c) = p_2(T_c)$

The derivatives of p_1 and p_2 may not be equal at T_c. It is not possible that all derivatives are equal of course, because $p_1(T) \neq p_2(T)$

The order of phase transition is the first derivative that is not equal:

$$\frac{d^n p_1}{dT^n}(T_c) \neq \frac{d^n p_2}{dT^n}(T_c)$$

In the figure before we have a first-order phase transition.

Suppose now that the pressure of the fluid is

$$p = p(T, \vartheta)$$

where the further parameter ϑ is, e.g., the fraction of mass in phase 1.

Then:

$$\dot{S} = -s\frac{\dot{T}}{T} + \frac{a^3}{T}\dot{p} = -s\frac{\dot{T}}{T} + a^3\frac{\partial p}{\partial T}\frac{\dot{T}}{T} + a^3\frac{\partial p}{\partial \vartheta}\frac{\dot{\vartheta}}{T}$$

$$= a^3\frac{\partial p}{\partial \vartheta}\frac{\dot{\vartheta}}{T}$$

If the phase transition occurs at equilibrium, when $T = T_c$, $p_1 = p_2$ and $\frac{dp}{d\vartheta} = 0$

So to create entropy one needs a phase transition of first order that takes place out of equilibrium, after a period of "supercooling".