Lecture 19 — Quantum fields in an expanding Universe

Vittorio, chap. 4 ; Carroll, 159-165

It is possible to obtain GR from a least-action principle.

As a Lagrangian density we use:

$$\mathcal{L} = R - 2\Lambda$$

The action (Einstein-Hilbert action) is:

$$S_H = \int d^4x \, (R - 2\Lambda)\sqrt{-g}$$

The dynamical field is the metric $g_{\mu\nu}$; however, given that $g_{\mu\nu;\alpha} = 0$ we cannot minimise the action using Euler-Lagrange equations.

Let's perturb the (inverse) metric $g^{\mu\nu}$, taking into account that:

- a manifold can have many metrics,
- a connection Γ is associated to each metric,
- the perturbation of a connection $\delta\Gamma$ is a tensor,
- Einstein equations will determine which metric describes gravity

The perturbation of S_H can be written as ($\Lambda = 0$)

$$\delta S = \int d^4x \sqrt{-g}\, g^{\mu\nu} \delta R_{\mu\nu} + \int d^4x \sqrt{-g}\, R_{\mu\nu} \delta g^{\mu\nu} + \int d^4x \, \delta\sqrt{-g}\, g^{\mu\nu} R_{\mu\nu}$$

The first term can be written as the integral of a divergence, so it becomes the flux of a perturbation over a very large surface, that can be set to zero.

$\delta\sqrt{-g}$:

we use the following properties of the metric:

$$g_{,\alpha} = g\, g^{\mu\nu} g_{\mu\nu,\alpha}$$

if $g^{\mu\nu} \to g^{\mu\nu} + \delta g^{\mu\nu} \Rightarrow \delta g_{\mu\nu} = -g_{\mu\alpha} g_{\nu\beta} \delta g^{\alpha\beta}$

then:

$$\delta g = g\, g^{\mu\nu} \delta g_{\mu\nu} = -g\, g_{\mu\nu} \delta g^{\mu\nu}$$

$$\delta\sqrt{-g} = \frac{1}{2\sqrt{-g}}\delta g = -\frac{1}{2} g_{\mu\nu} \delta g^{\mu\nu} \sqrt{-g}$$

Then:

$$\delta S = \int d^4x \sqrt{-g}\left[R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\right] \delta g^{\mu\nu}$$

$$\delta S = 0 \quad \Rightarrow \quad R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = G_{\mu\nu} = 0$$

Adding Λ, we add a term to δS :

$$\int d^4x\, \delta\sqrt{-g}\,(-2\Lambda) = \int d^4x \sqrt{-g}\, \Lambda g_{\mu\nu} \delta g^{\mu\nu}$$

$$\delta S = 0 \quad \Rightarrow \quad R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0$$

Now we add "matter", meaning something that is not geometry

$$S = \frac{S_H}{16\pi G} + S_M$$

Let's define : $\quad T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\partial S_M}{\partial g^{\mu\nu}}$

$$\Rightarrow R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

We write the action of matter as that of a quantum field ϕ

$$S_M = \int d^4x \sqrt{-g}\left[-\frac{1}{2} g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - V(\phi)\right]$$

where $V(\phi)$ is an effective potential for the field.

$$\delta S_M = \int d^4x\, \delta\sqrt{-g}\left[-\frac{1}{2} g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - V(\phi)\right] - \frac{1}{2}\int d^4x \sqrt{-g}\,\phi_{,\alpha}\phi_{,\beta}\,\delta g^{\alpha\beta}$$

$$= \int d^4x \sqrt{-g}\left[\frac{1}{4} g_{\mu\nu} g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} + \frac{1}{2} g_{\mu\nu} V(\phi) - \frac{1}{2}\phi_{,\mu}\phi_{,\nu}\right]\delta g^{\mu\nu}$$

then:

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - g_{\mu\nu} V(\phi)$$

In a FLRW universe ϕ can only depend on time:

$$\phi_{,\mu} \longrightarrow (\dot\phi, 0, 0, 0)$$

$$T_{00} = \dot\phi^2 - \frac{1}{2}(-1)(-\dot\phi^2) + V(\phi) = \frac{1}{2}\dot\phi^2 + V(\phi) = \rho$$

$$T_{ij} = -\frac{1}{2} g_{ij}(-\dot\phi^2) - g_{ij} V(\phi) = g_{ij}\left(\frac{1}{2}\dot\phi^2 - V(\phi)\right) =$$

$$= g_{ij}\, P$$

$$\Longrightarrow \quad \rho = \frac{1}{2}\dot\phi^2 + V(\phi)$$

$$P = \frac{1}{2}\dot\phi^2 - V(\phi)$$

The dynamics of the field is given by $T^{\mu\nu}{}_{;\mu} = 0$ that gives a Klein-Gordon equation:

$$\phi^{;\mu}{}_{;\mu} - \frac{dV}{d\phi} = 0$$

It is easier to use third Friedmann equation:

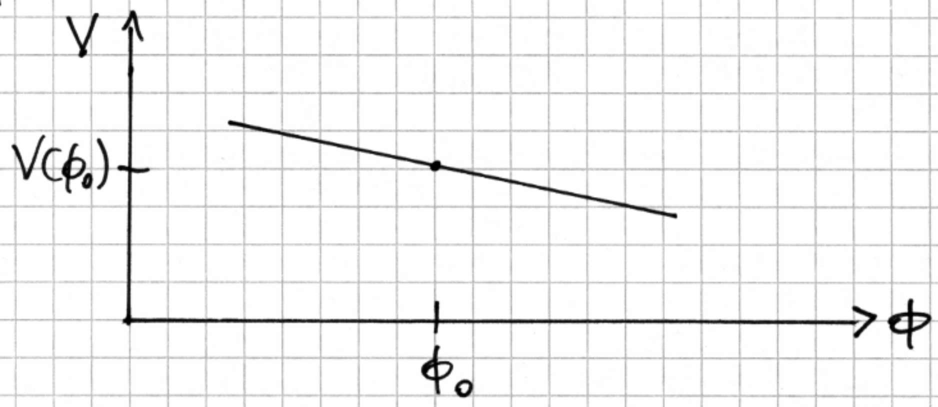
$$\dot\rho + 3\frac{\dot a}{a}(P + \rho) = 0$$

$$\dot\rho = \dot\phi\ddot\phi + \frac{dV}{d\phi}\dot\phi \quad , \quad p+\rho = \dot\phi^2$$

$$\Rightarrow \quad \ddot\phi + 3\frac{\dot a}{a}\dot\phi + \frac{dV}{d\phi} = 0$$

This is the equation of motion of a mass with friction $3H$ moving in a potential $V(\phi)$, subject to a force $-\frac{dV}{d\phi}$

Suppose the field ϕ is having a phase transition, it is out of equilibrium and subject to a shallow potential:



The induced "slow rolling" is slowed down by friction

$$\Rightarrow \dot\phi^2 \ll V(\phi)$$

This results in an equation of state:

$$p = w\rho \quad \Rightarrow \quad w = \frac{\frac{1}{2}\dot\phi^2 - V(\phi)}{\frac{1}{2}\dot\phi^2 + V(\phi)} \approx -1$$

$$\rho \simeq V(\phi), \quad p \simeq -V(\phi)$$

So a quantum field slowly rolling down a potential enters like a cosmological constant

Inflation ends when the field gets into the new vacuum configuration.

$\Rightarrow$ oscillations around the vacuum state

$\Rightarrow$ decay to all possible particles

$\Rightarrow$ reheating

In other words, because the field does not oscillate, its potential $V(\phi)$ gives a constant energy density and behaves as a Λ

Can we use this mechanism to explain our low-energy Λ?

Let's give an energy $\frac{1}{2}h\nu$ to each oscillation mode of a field:

$$\rho c^2 = \frac{1}{(2\pi\hbar)^3}\int_0^\infty \frac{1}{2}h\nu\, d^3p = \frac{4\pi c}{2(2\pi\hbar)^3}\int_0^\infty p^3\, dp$$

where $h\nu = pc$, and we neglected mass.

The integral diverges.

Let's truncate the integral to a scale mc:

$$\rho c^2 = \frac{1}{4\pi^2}\frac{c}{\hbar^3}\int_0^{mc} p^3 dp = \frac{1}{16\pi^2}\frac{m^4c^5}{\hbar^3}$$

If $m = m_P$: $\quad \rho c^2 = \frac{1}{16\pi^2} E_P = \frac{(10^{19})^4}{16\pi^2} \text{GeV}^4 =$

$$= 10^{74}\ \text{GeV}^4$$

This may work around or below Planck time, when inflation is expected to happen, but at t_0!

$$\rho_P = 5.2\times 10^{93}\ \text{g cm}^{-3}$$

$$\rho_\Lambda = 6.4\times 10^{-30}\ \text{g cm}^{-3}$$

$$\frac{1}{16\pi^2}\frac{\rho_P}{\rho_\Lambda} \sim 4\times 10^{120}$$

One can revert the expression for the cutoff mass as:

$$m_\Lambda = \left(\frac{16\pi^2\hbar^3}{c^5}\rho_\Lambda\right)^{1/4} \simeq 8\times 10^{-3}\ \text{eV} \ll m_P$$

Such a tiny cutoff mass is very hard to explain.