Lecture 1

Einstein equations
Schutz, chapters 7, 8

BASIC PRINCIPLES

(1) Spacetime is a 4-dimensional manifold
+ differentiable as many times as needed
+ pseudo-Riemaniann

$g_{\alpha\beta}$: METRIC $(-+++)$

(2) The metric is measurable by rods and clocks
→ distance interval: $dl = \sqrt{g_{\alpha\beta}dx^\alpha dx^\beta}$, x^α space-l
→ time interval: $dt = \sqrt{-g_{\alpha\beta}dx^\alpha dx^\beta}$, x^α time-like

A frame where $g_{\alpha\beta} = \eta_{\alpha\beta}$ everywhere does not exist

(3) For (almost) any event P we can find a coordinate system such that, locally:

$$g_{\alpha\beta}(P) = \eta_{\alpha\beta}(P) \quad \text{and} \quad g_{\alpha\beta,\gamma}(P) = 0$$

LOCAL INERTIAL FRAME $(\eta_{\alpha\beta} = \mathrm{diag}(-1,1,1,1))$

(4) Freely falling particles move along geodesics

$$\frac{d^2x^\alpha}{d\tau} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

→ if $g_{\alpha\beta,\gamma} = 0$ then $\Gamma^\alpha_{\mu\nu} = 0 \Rightarrow \frac{d^2x^\alpha}{d\tau^2} = 0$

WEAK EQUIVALENCE PRINCIPLE

(5) STRONG EQUIVALENCE PRINCIPLE:

Any law valid in Special Relativity has the same expression in a local inertial frame

⇒ comma to semicolon rule

This is not obvious: e.g. particle conservation $(nU^\alpha)_{,\alpha} = 0$ in SR may become $(nU^\alpha)_{;\alpha} = \text{const} \cdot R$ in GR

⇒ minimal coupling to the metric

EINSTEIN EQUATIONS

$G^{\alpha\beta} \equiv R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta}$ has the property: $G^{\alpha\beta}{}_{;\alpha} = 0$

$T^{\alpha\beta}{}_{;\alpha} = 0$ is energy-momentum conservation, so

$G^{\alpha\beta} = k\, T^{\alpha\beta}$ is Einstein's guess

Adding Λ ($g^{\alpha\beta}{}_{;\alpha} = 0$ as well):

$$\boxed{R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta} + \Lambda g^{\alpha\beta} = 8\pi G T^{\alpha\beta}} \qquad \left(c \neq 1 : \frac{8\pi G}{c^4}\right)$$

giving:
- \+ weak field limit, Newtonian physics
- \+ post-Newtonian expansion, precession of perihelia
- \+ gravitational lensing
- \+ strong field regime, black holes
- \+ gravitational waves
- \+ cosmology...

Absorbing Λ in the stress-energy tensor

$$G^{\alpha\beta} = 8\pi G \left(T^{\alpha\beta} - \frac{\Lambda}{8\pi G} g^{\alpha\beta} \right)$$

For a perfect fluid: $T^{\alpha\beta} = (p+\rho) U^\alpha U^\beta + p g^{\alpha\beta}$

in a local inertial frame: $T^{\alpha\beta} = \mathrm{diag}(\rho, p, p, p)$

call: $\rho_\Lambda = \frac{\Lambda}{8\pi G} \Rightarrow -\frac{\Lambda}{8\pi G} g^{\alpha\beta} = \mathrm{diag}(\rho_\Lambda, -\rho_\Lambda, -\rho_\Lambda, -\rho_\Lambda)$

This is the stress-energy tensor of a perfect fluid with a weird equation of state:

$$p = -\rho$$

In the following we will often "hide" Λ in $T^{\alpha\beta}$

An equivalent expression of Einstein equations

Trace: $R^\alpha{}_\alpha - \frac{1}{2} R \delta^\alpha_\alpha = 8\pi G T^\alpha{}_\alpha$, $\delta^\alpha_\alpha = 4$

call $T \equiv T^\alpha{}_\alpha$, $R - 2R = -R = 8\pi G T$

$$\Rightarrow \boxed{R^{\alpha\beta} = 8\pi G \left(T^{\alpha\beta} - \frac{1}{2} T g^{\alpha\beta} \right)}$$

This is very useful for our calculations

CONSERVED QUANTITIES for a freely falling particl
(Sec. 7.4)

Take a particle traveling along a time-like or null geodesi
We know that:

$$p^\alpha = m U^\alpha, \quad U^\alpha = \frac{dx^\alpha}{d\tau}, \quad \frac{d}{d\tau} = \frac{dx^\alpha}{d\tau}\frac{\partial}{\partial x^\alpha} = U^\alpha \frac{\partial}{\partial x^\alpha}$$

Take the geodesic equation obtained by transportin
a one-form along its dual vector:

$$\frac{d^2 x_\alpha}{d\tau^2} - \Gamma^\mu{}_{\nu\alpha} \frac{dx_\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \qquad \times m^2$$

$$m^2 \frac{d^2 x_\alpha}{d\tau} = m^2 U^\mu U_{\alpha,\mu} = p^\mu p_{\alpha,\mu}$$

$$p^\mu p_{\alpha,\mu} = \Gamma^\mu{}_{\nu\alpha} p_\mu p^\nu = \frac{1}{2} g^{\mu\beta}\left(g_{\beta\nu,\alpha} + g_{\beta\alpha,\nu} - g_{\alpha\nu,\beta}\right) p_\mu p^\nu$$

$$= \frac{1}{2} p^\beta p^\nu g_{\beta\nu,\alpha} + \frac{1}{2} p^\beta p^\nu \left(g_{\beta\alpha,\nu} - g_{\nu\alpha,\beta}\right)$$

where we have used $g^{\mu\beta}$ to raise the index of p_μ.

The second term is a contraction of a symmetric tensor $(p^\beta p^\nu)$ and an antisymmetric tensor $(g_{\beta\alpha,\nu} - g_{\nu\alpha,\beta})$, so it vanishes, and

$$p^\mu p_{\alpha,\mu} = m \frac{dp_\alpha}{d\tau} = \frac{1}{2} p^\mu p^\nu g_{\mu\nu,\alpha}$$

It follows that, in a coordinate system, the coefficients of the metric do not depend on α coordinate, then $g_{\mu\nu,\alpha} = 0$ for that α and:

p_α IS CONSERVED ALONG ITS TRAJECTORY

NB: this applies to the momentum one-form p_α, not to the momentum vector p^α