Lecture 2 Schwartzschild metric

Schutz, Chapters 10-11
Carrol', pagg. 201-204

The aim is to solve Einstein equations in the case of static spherical symmetry with a point mass at the origin of the coordinate system

SPHERICAL SYMMETRY FOR THE METRIC

→ In Minkowski space and spherical coordinates:

$$ds^2 = -dt^2 + dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)$$

Take a surface at $t = \text{const}$, $r = \text{const}$, a length along this two-sphere is:

$$dl^2 = r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2) = r^2 d\Omega^2$$

→ Spherical symmetry: we can find a coordinate system such that every event that lies on a two-sphere ($r=\text{const}$, $t=\text{const}$) has a line element of the form:

$$dl^2 = f^2(r,t)\, d\Omega^2$$

changing coordinates, $r = f(r', t)$:

$$\Rightarrow dl^2 = r^2 d\Omega^2$$

→ Take $t = \text{const}$, and take two concentric spheres at r and $r+dr$; we can rotate their ϑ and φ coordinates so that:

$$\vec{e}_r \cdot \vec{e}_\vartheta = \vec{e}_r \cdot \vec{e}_\varphi = 0$$

while the line element guarantees that $\vec{e}_\vartheta \cdot \vec{e}_\varphi = 0$

→ The same reasoning applies to two spheres with the same r at t and $t+dt$:

$$\vec{e}_t \cdot \vec{e}_\vartheta = \vec{e}_t \cdot \vec{e}_\varphi = 0$$

→ $\vec{e}_t \cdot \vec{e}_r$ can be set to 0 by a suitable change of coordinates (Carroll) or by asking time-reversal symmetry (Schutz):

$$\vec{e}_t \cdot \vec{e}_r = 0$$

We can write the metric as:

$$ds^2 = -e^{2\phi} dt^2 + e^{2\Lambda} dr^2 + r^2 d\Omega^2$$

$\phi(r)$ and $\Lambda(r)$ are generic functions, $\Lambda(r)$ is not the cosmological constant!

At $r \to \infty$ we should recover SR (Minkowski):

$$\phi(r), \Lambda(r) \xrightarrow[r\to\infty]{} 0$$

We now solve Einstein equations with this metric:

$$g_{\mu\nu} = \mathrm{diag}\left(-e^{2\phi}, e^{2\Lambda}, r^2, r^2 \mathrm{sen}^2\vartheta\right)$$

The steps are (see textbooks)

→ compute inverse metric $g^{\mu\nu}$

→ compute metric derivatives $g_{\mu\nu,\alpha}$

→ compute Christoffel symbols $\Gamma^\alpha{}_{\mu\nu}$

To compute the Ricci tensor one can adopt two strategies

- compute all non-vanishing terms of $R^\alpha{}_{\beta\mu\nu}$ and then contract
- contract the expression for the Riemann tensor, compute $\Gamma^\mu{}_{\alpha\mu}$, $\Gamma^\mu{}_{\alpha\mu,\beta}$, $\Gamma^\sigma{}_{\mu\alpha}\Gamma^\mu{}_{\sigma\beta}$ then go to the Ricci tensor

Ricci tensor is diagonal its components result:

$$R_{00} = e^{2(\phi-\Lambda)}\left[\phi_{,r,r} + (\phi_{,r})^2 - \phi_{,r}\Lambda_{,r} + \frac{2}{r}\phi_{,r}\right]$$

$$R_{11} = -\phi_{,r,r} - (\phi_{,r})^2 + \phi_{,r}\Lambda_{,r} + \frac{2}{r}\Lambda_{,r}$$

$$R_{22} = e^{-2\Lambda}\left(r\Lambda_{,r} - r\phi_{,r} - 1\right) + 1$$

$$R_{33} = \mathrm{sen}^2\vartheta\, R_{22}$$

Ricci scalar:

$$R = g^{\alpha\beta}R_{\alpha\beta} = -2e^{-2\Lambda}\left[\phi_{,r,r} + (\phi_{,r})^2 - \Lambda_{,r}\phi_{,r} + \frac{2}{r}\phi_{,r} - \frac{2}{r}\Lambda_{,r} + \frac{1}{r^2}\left(1 - e^{2\Lambda}\right)\right]$$

Point mass: $T^{\alpha\beta} = 0$ everywhere but at the origin

$\Rightarrow$ VOID SOLUTION: $R_{\alpha\beta} = 0$

00 component: $\frac{2}{r}\phi_{,r} = -\phi_{,rr} - (\phi_{,r})^2 + \phi_{,r}\Lambda_{,r}$

11 component: $\frac{2}{r}\phi_{,r} + \frac{2}{r}\Lambda_{,r} = 0 \Rightarrow \phi_{,r} = -\Lambda_{,r}$

$$\Rightarrow \phi = -\Lambda + \text{const}$$

constant: $\phi \xrightarrow[r\to\infty]{} 0$, $\Lambda \xrightarrow[r\to\infty]{} 0 \Rightarrow \text{const} = 0$

$$\Rightarrow \phi = -\Lambda$$

22 component: $e^{-2\Lambda} - 2r\Lambda_{,r}e^{-2\Lambda} = 1$

$$\Rightarrow (r e^{-2\Lambda})_{,r} = 1$$

$$\Rightarrow r e^{-2\Lambda} = r + \text{const}$$

$$\Rightarrow e^{-2\Lambda} = 1 + \frac{\text{const}}{r}$$

constant: we want to recover the weak field limit

$$e^{2\Lambda} \xrightarrow[r \text{ large}]{} 1 - 2\Phi, \quad \Phi = -\frac{GM}{r}$$ grav. potential

where M is the mass of the singularity

$$\Rightarrow e^{-2\Lambda} = 1 + \frac{\text{const}}{r} \longrightarrow 1 + 2\left(-\frac{GM}{r}\right)$$

$$\Rightarrow \text{const} = -2GM$$

$$\Rightarrow \boxed{ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \frac{1}{1 - \frac{2GM}{r}}dr^2 + r^2 d\Omega^2}$$

$\longrightarrow$ This metric is singular at:

$$r = 0, \quad r = 2GM = R_s \quad \left(= \frac{2GM}{c^2}\right)$$

Schwartzschild radius

$\rightarrow$ Laplace / Mitchell dark star:

$$V_{esc}^2 = \frac{2GM}{R} = c^2 \implies R = \frac{2GM}{c^2} = R_s$$

$\rightarrow$ Geometrical units: $G = 1$

$$\implies R_s = 2M$$

$\rightarrow$ GRAVITATIONAL RADIUS:

$$R = \frac{GM}{c^2}, \quad \text{or} \quad R = GM \ (c=1) \quad \text{or} \quad R = M \ (G=1)$$

MOTION OF A MASSIVE PARTICLE AROUND A BH

Schwartzschild metric does not depend on t and φ

$\implies p_0$ and p_3 are conserved (p_3 alias p_φ)

$$\tilde{E} \equiv -\frac{p_0}{m}, \quad \tilde{L} \equiv \frac{p_\varphi}{m}$$

If we restrict the orbit to a plane we can set

$$\vartheta = \frac{\pi}{2} \implies \frac{d\vartheta}{d\tau} = 0$$

Let's compute the orbit using $\vec{p} \cdot \vec{p} = -m^2$

$$p_0 = -m\tilde{E} \qquad p^0 = g^{00} p_0 = \frac{m\tilde{E}}{1 - \frac{2GM}{r}}$$

$$p_r = g_{rr} p^r = \frac{m}{1 - \frac{2GM}{r}} \frac{dr}{d\tau} \qquad p^r = m \frac{dr}{d\tau}$$

$$p_\vartheta = 0 \qquad p^\vartheta = 0$$

$$p_\varphi = m\tilde{L} \qquad p^\varphi = g^{\varphi\varphi} p_\varphi = \frac{m\tilde{L}}{r^2}$$

$$\vec{p} \cdot \vec{p} = p^\alpha p_\alpha = -m^2 = \frac{-m^2 \tilde{E}^2}{1 - \frac{2GM}{r}} + \frac{m^2}{1 - \frac{2GM}{r}} \left(\frac{dr}{d\tau}\right)^2 + \frac{m^2 \tilde{L}^2}{r^2}$$

$$\implies \left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - \left(1 - \frac{2GM}{r}\right)\left(1 + \frac{\tilde{L}^2}{r^2}\right) = \tilde{E}^2 - \tilde{V}^2(r)$$

EFFECTIVE POTENTIAL:

$$\tilde{V}^2(r) = \left(1 - \frac{2GM}{r}\right)\left(1 + \frac{\tilde{L}^2}{r^2}\right)$$

$r \to \infty$: $\tilde{V}^2(r) \to 1$, $r = 2GM$: $V^2(r) = 0$

Newtonian case:

centrifugal barrier, h is the specific angular momentum

$$V_{eff}(r) = -\frac{GM}{r} + \frac{h^2}{2r^2}$$

the centrifugal barrier makes it impossible to fall onto the point mass, unless $h=0$ exactly

GR: we can find the extrema of $\tilde{V}^2(r)$ by setting $\frac{d\tilde{V}^2}{dr} = 0$ $\Rightarrow$ two solutions if

$$\tilde{L}^2 > 12(GM)^2 \Rightarrow r = \frac{\tilde{L}^2}{2GM}\left(1 \pm \sqrt{1 - \frac{12(GM)^2}{\tilde{L}^2}}\right)$$

For $\tilde{L}^2 = 12(GM)^2$ we have the

INNERMOST STABLE ORBIT (Last stable orbit)

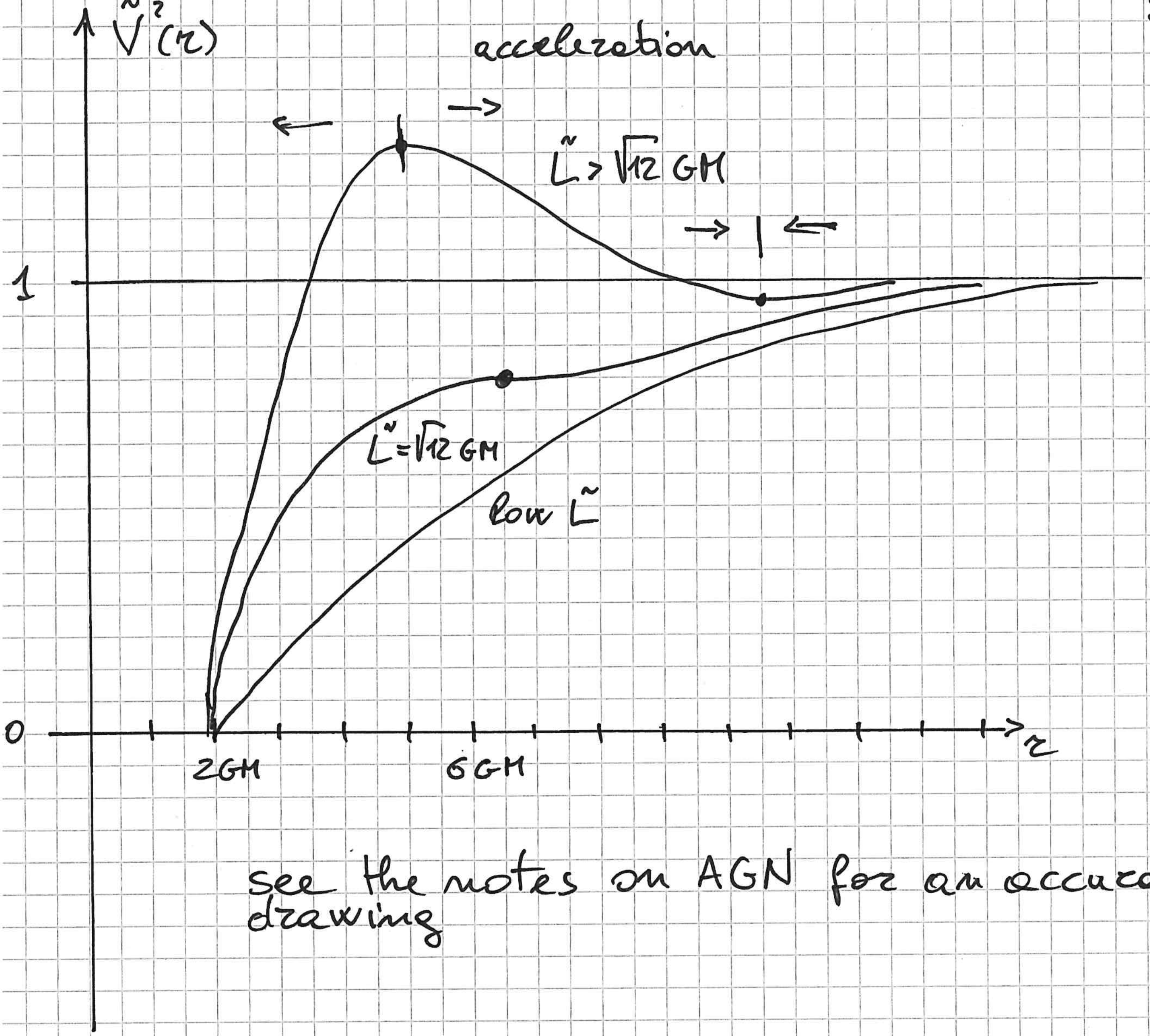
$$r = 6\,GM = 3\,R_s$$

Acceleration:

$$\frac{d}{d\tau}\left[\left(\frac{dr}{d\tau}\right)^2\right] = 2\frac{dr}{d\tau}\frac{d^2r}{d\tau^2} = \frac{d}{d\tau}\left(\tilde{E}^2 - \tilde{V}^2(r)\right) = -\frac{d\tilde{V}^2}{dr}\frac{dr}{d\tau}$$

$$\Rightarrow \frac{d^2r}{d\tau^2} = -\frac{1}{2}\frac{d}{dr}\tilde{V}^2(r)$$

so the "gravitational force" is directed where the potential decreases



see the notes on AGN for an accurate drawing