Lecture 3 — The event horizon

Schutz, Chap. 11

The Schwartzschild metric has two singularities, $r=0$ (point mass) and $r=2GM$

$r=2GM$:

→ it can be locally removed by a coordinate transformatio (see Schutz)

→ curvature does not diverge ($R_{\mu\nu}=0$)

→ tides are strong, especially for a stellar mass BH

$r<2GM$:

→ g_{00} and g_{rr} change sign!

→ ALL TIME-LIKE AND NULL GEODESICS HAVE THE $r=0$ SINGULARITY IN THEIR FUTURE LIGHT CONE

This implies that no particle can escape the BH if it starts from $r<2GM$, even light

⇒ EVENT HORIZON

Take a particle falling on a radial orbit: $\tilde{L}=0$

$$\left(\frac{dr}{d\tau}\right)^2 = \tilde{E} - 1 + \frac{2GM}{r} \qquad \text{NB: } dr<0$$

Proper time to infall from $r_0>2GM$ to $2GM$:

$$\Delta\tau = \int d\tau = -\int_{r_0}^{2GM} \frac{d\tau}{dr}\,dr = -\int_{r_0}^{2GM} \frac{dr}{\sqrt{\tilde{E}^2 - 1 + \frac{2GM}{r}}}$$

this integral does not diverge for $\tilde{E}^2>0$

⇒ a particle will cross the event horizon after a finite time of its clock

Coordinate time to infall from $r_0>2GM$ to $2GM$:

$$\Delta t = \int dt = -\int_{r_0}^{2GM} \frac{dt}{d\tau}\frac{d\tau}{dr}\,dr$$

where $\frac{dt}{d\tau} = \frac{1}{U^0}$, $U^0 = \frac{P}{m} = \frac{\tilde{E}}{1-\frac{2GM}{r}}$

$$\Rightarrow \Delta t = -\int_{r_0}^{2GM} \frac{\tilde{E}\,dr}{\left(1-\frac{2GM}{r}\right)\sqrt{\tilde{E}^2 - 1 + \frac{2GM}{r}}}$$

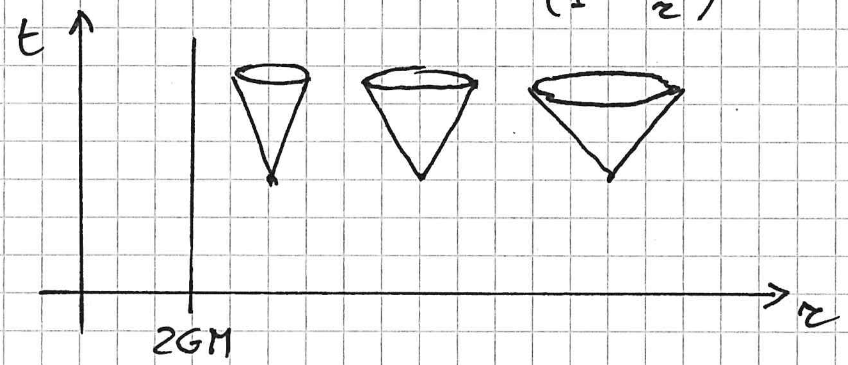
This integral has a logarithmic divergence

$\Rightarrow$ a distant observer will never see the particle cross the event horizon

FUTURE LIGHT CONE

Take a photon emitted on a forward radial orbit, it travels along a null geodesic

$$ds^2 = 0 \quad \Rightarrow \quad dt^2 = \frac{1}{\left(1-\frac{2GM}{r}\right)^2} dr^2 \,, \quad \frac{dt}{dr} = \frac{1}{1-\frac{2GM}{r}}$$



Approaching the event horizon, the null lines of the future light cone get steeper, and close at $r = 2GM$

GRAVITATIONAL REDSHIFT

Take:
- + a body at rest at $r \gtrsim 2GM$
- + a photon emitted by this body in forward radial orbit
- + an observer at $r \to \infty$ that receive the photon

We know that in a frame with 4-velocity $\vec{U}$ a particle of momentum $\vec{p}$ is seen with energy

$$E = -\vec{U}\cdot\vec{p}$$

Indeed, in the (free-falling) momentarily comoving reference frame of the particle, $\vec{U} = (1,0,0,0)$ and $\vec{p} = (E, p_x, p_y, p_z)$, so $\vec{U}\cdot\vec{p} = -E$, and it is covariant.

This properties can be extended to any frame with velocity $\vec{U}$

What is $\vec{U}$ for the body? only U^0 can be $\neq 0$, and

$$\vec{U}\cdot\vec{U} = -1 = g_{00}\, U^0 U^0 \quad \Rightarrow \quad U^0 = \sqrt{-\frac{1}{g_{00}}} = \frac{1}{\sqrt{1-\frac{2GM}{r}}}$$

So the energy of the photon seen by the body, the emission energy, is:

$$E_{em} = -U^{\alpha} p_{\alpha} = -\frac{p_0}{\sqrt{1-\frac{2GM}{r}}} = h\nu_{em}$$

For the distant observer $\vec{U} = (1,0,0,0)$, so

$$E_{obs} = -U^{\alpha} p_{\alpha} = -p_0 = h\nu_{obs}$$

p_0 is conserved, so:

$$h\nu_{em} = \frac{h\nu_{obs}}{\sqrt{1-\frac{2GM}{r}}}$$

Radiation is thus shifted to longer wavelengths - redshifted

$$z \equiv \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}} = \frac{\nu_{em}}{\nu_{obs}} - 1 = \frac{1}{\sqrt{1-\frac{2GM}{r}}} - 1$$

At $r = 2GM$, $z \to \infty$, photons emitted at the event horizon are redshifted to zero energy

Now suppose: $r \gg \frac{2GM}{c^2}$, $\sqrt{1-\frac{2GM}{rc^2}} \simeq 1 - \frac{GM}{rc^2}$

$$h\nu_{obs} \cong h\nu_{em} - \frac{GM}{r}\frac{h\nu_{em}}{c^2}$$

as if $h\nu_{em}/c^2$ is the effective mass of the photon, that has lost energy by climbing the potential well