Lecture 4

Photon capture radius

Schutz, Chap 11

Let's compute the trajectory of a photon near a black hole:

p_0 and p_φ are constant, $\tilde{E} \equiv -p_0$, $\tilde{L} \equiv p_\varphi$

We know that $E_\infty = h\nu_{obs} = -p_0 = \tilde{E}$

Assume: $\theta = \frac{\pi}{2}$, $p^\theta = 0$, λ is the affine parameter of geodesic

$$p_0 = -\tilde{E}, \qquad p^0 = \frac{\tilde{E}}{1-\frac{2GM}{r}}$$

$$p_r = \frac{1}{1-\frac{2GM}{r}}\frac{dr}{d\lambda}, \qquad p^r = \frac{dr}{d\lambda}$$

$$p_\varphi = \tilde{L}, \qquad p^\varphi = \frac{\tilde{L}}{r^2}$$

$$\vec{p}\cdot\vec{p} = 0 = -\frac{\tilde{E}^2}{1-\frac{2GM}{r}} + \left(\frac{dr}{d\lambda}\right)^2 \frac{1}{1-\frac{2GM}{r}} + \frac{\tilde{L}^2}{r^2}$$

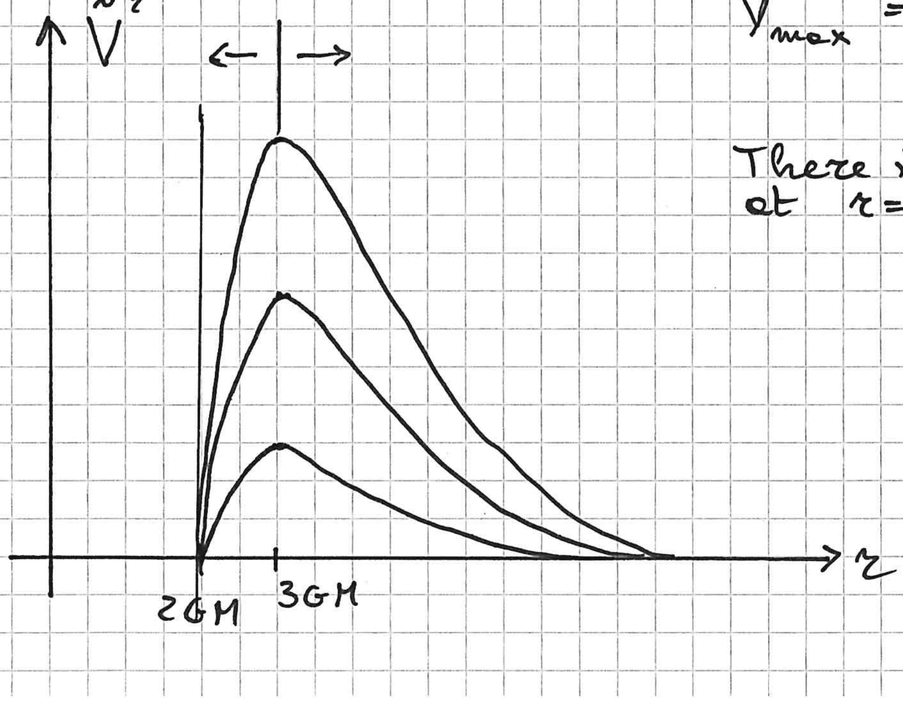
$$\Rightarrow \left(\frac{dr}{d\lambda}\right)^2 = \tilde{E}^2 - \left(1-\frac{2GM}{r}\right)\frac{\tilde{L}^2}{r^2} = \tilde{E}^2 - \tilde{V}^2(r)$$

$$\tilde{V}^2(r) = \frac{\tilde{L}^2}{r^2}\left(1-\frac{2GM}{r}\right)$$

$r \to \infty$, $\tilde{V}^2 \to 0$
$r = 2GM$, $\tilde{V}^2 = 0$

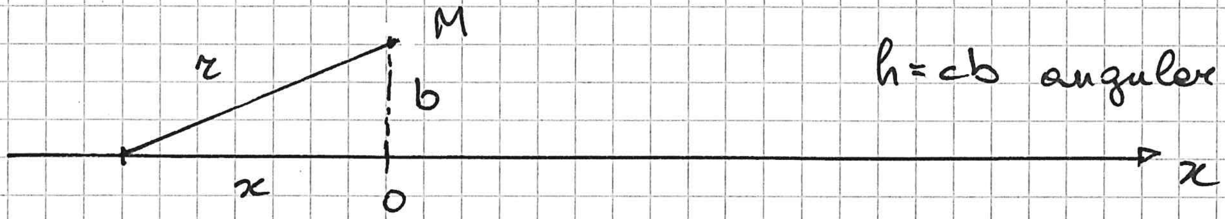
Extreme: $\frac{d\tilde{V}^2}{dr} = 0 \Rightarrow r = 3GM$ independent of $\tilde{L}$

$$\tilde{V}^2_{max} = \frac{\tilde{L}^2}{27(GM)^2}$$



There is an unstable orbit at $r = 3GM$

Photon trajectory in Newtonian dynamics



$h = cb$ angular momentum

$$x = ct\,, \quad r^2 = b^2 + x^2 = b^2 + c^2t^2 \quad \Rightarrow \quad 2r\frac{dr}{dt} = 2c^2 t$$

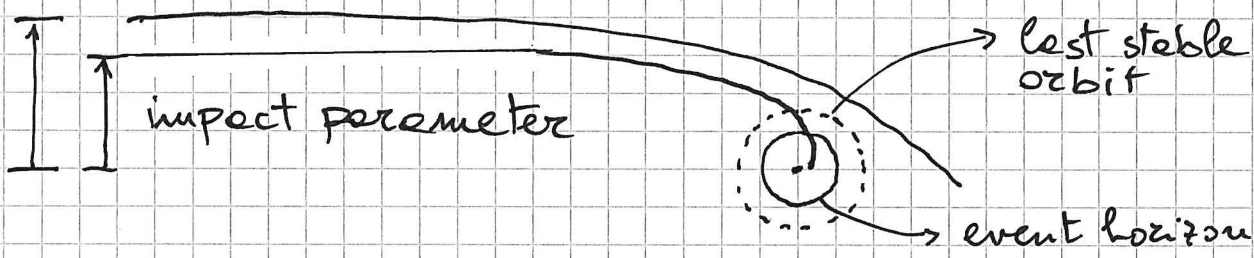
$$\Rightarrow \left(\frac{dr}{dt}\right)^2 = c^2\left(1 - \frac{b^2}{r^2}\right)\,, \quad V_{eff}(r) = \frac{c^2b^2}{r^2} = \frac{h^2}{r^2}$$

the straight path of the photon implies a centrifugal barrier

→ Impact parameter for a particle around a black hole at infinity:

$$\tilde{L} = pb = h\,, \quad \tilde{E} = p \quad \Rightarrow \quad b = \frac{\tilde{L}}{\tilde{E}} = \frac{\tilde{L}}{E_\infty}$$

→ Capture radius of a black hole:



Impact parameter is what is seen from a distant observer.
Photon trajectory:

$$\left(\frac{dr}{d\lambda}\right)^2 = E_\infty^2\left[1 - \left(1 - \frac{2GM}{r}\right)\frac{b^2}{r^2}\right]$$

$$\tilde{V}^2(r) = E_\infty^2\left(1 - \frac{2GM}{r}\right)\frac{b^2}{r^2}$$

Photon capture radius: the smallest value of b before the photon falls on the BH

→ Orbit pericenter: $\frac{dr}{d\varphi} = 0$

$$\frac{dr}{d\varphi} = \frac{dr}{d\lambda}\frac{d\lambda}{d\varphi} = \frac{p^r}{p^\varphi}\,, \quad p^\varphi = \frac{bE_\infty}{r^2}$$

$$\left(\frac{dr}{d\varphi}\right)^2 = \frac{r^4}{b^2}\left[1 - \frac{b^2}{r^2}\left(1 - \frac{2GM}{r}\right)\right] = 0$$

$$\Rightarrow b^2(r_{peri} - 2GM) = r_{peri}^3$$

but photons with $r_{peri} \leq 3GM$ are lost

$$b^2 \geq \frac{(3GM)^3}{(3GM - 2GM)} = 27\,(GM)^3$$

This defines the photon capture radius:

$$b \geq \sqrt{27}\; GM$$