Lecture 6 — Robertson-Walker metric

Schutz, chapter 12
Vittorio, chapter 1

We start from:

(1) the cosmological principle, anchored to the observed isotropy of the CMB

(2) Hubble-Lemaitre's evidence of Universe expansion

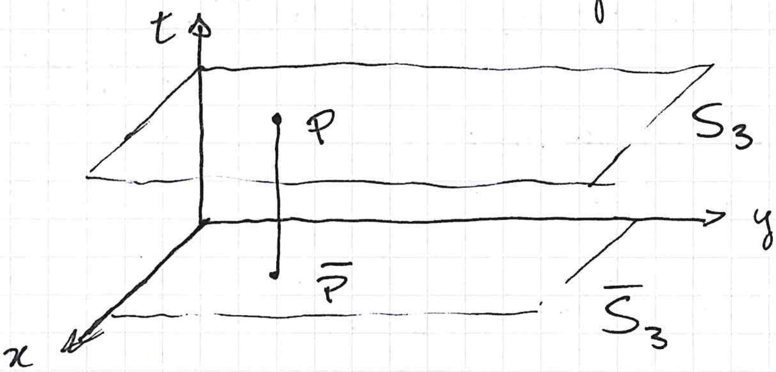
→ Spacetime can be sliced into hypersurfaces of constant time that are homogeneous and isotropic

→ the mean rest frame of galaxies agrees with this definition of simultaneity

NB: this implies that in cosmology there is a "preferred" frame, that for which the CMB is isotropic

But this DOES NOT contradict SR, this frame is comoving with radiation, the laws of physics are valid in any frame.

Let's construct a synchronous (frame) gauge:



Let's consider:

- $\bar{S}_3$ at $x^0 = \bar{x}^0$
- $\bar{P}$ at $\bar{x}^k$
- S_3 at $x^0 = \bar{x}^0 + dx^0$
- a time-like geodesic passing through $\bar{P}$

tangent vector:

$$n^\alpha = \frac{dx^\alpha}{d\tau} = (1,0,0,0)$$

If $\bar{V}^\alpha = \{0, \bar{V}^k\}$ is on $\bar{S}^3$, $n_\alpha \bar{V}^\alpha = g_{\alpha\beta} n^\alpha \bar{V}^\beta = g_{0k} \bar{V}^k = 0$

$\Rightarrow g_{0k} = 0$ → Needed to synchronize clocks (appendix D.2, Vittorio)

This geodesic uniquely defines a point P on S_3

0-component of geodesic equation: $\frac{d^2x^0}{d\tau^2} + \Gamma^0_{00} = 0$

$$\Rightarrow \frac{d}{d\tau}\frac{dx^0}{d\tau} = \frac{d}{d\tau} 1 = -\frac{1}{2} g_{00,0} = 0 \Rightarrow g_{00} = \text{const} = -1$$

The metric is then written as:

$$ds^2 = -dt^2 + \mathcal{R}^2(t)\, h_{ij}\, dx^i dx^j = -dt^2 + dl^2$$

$\mathcal{R}(t)$: DIMENSIONAL SCALE FACTOR, it is a length
(Not to be confused with the Ricci scalar R !)

To compute h_{ij} we use the solution found for the case of spherical symmetry, imposing that the Ricci scalar is constant in space ($\mathcal{R}$=const=1, r is adimensional):

$$R = -2e^{-2\Lambda}\left[\phi'' + \phi'^2 - \Lambda'\phi' + \frac{2}{r}\phi' - \frac{2}{r}\Lambda' + \frac{1}{r^2}(1-e^{2\Lambda})\right]$$

We have : $g_{00} = -e^{2\phi} = -1 \Rightarrow \phi = 0$

$$\Rightarrow R = -2e^{-2\Lambda}\left[-\frac{2}{r}\Lambda' + \frac{1}{r^2}(1-e^{2\Lambda})\right] = \text{const} = 6k$$

$$= -\frac{2}{r^2}\left(-2r\Lambda' e^{-2\Lambda} + e^{-2\Lambda} - 1\right) =$$

$$= \frac{2}{r^2}\left[1 - \frac{d}{dr}(re^{-2\Lambda})\right] = \frac{2}{r^2}\frac{d}{dr}\left(r - re^{-2\Lambda}\right)$$

$$\Rightarrow 3kr^2 = \frac{d}{dr}\left[r\left(1-e^{-2\Lambda}\right)\right] \Rightarrow kr^3 + A = r(1-e^{-2\Lambda})$$

$$\Rightarrow e^{-2\Lambda} = 1 - kr^2 - \frac{A}{r} \Rightarrow g_{rr} = e^{2\Lambda} = \frac{1}{1-kr^2-\frac{A}{r}}$$

To fix A we require local flatness:

$$g_{rr}(r=0) = 1 \quad \Rightarrow A = 0$$

$$\Rightarrow g_{rr} = \frac{1}{1-kr^2}$$

$$dl^2 = \mathcal{R}^2\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right)$$

ROBERTSON - WALKER METRIC

→ r and k are adimensional, k can be rescaled to one of $k = -1, 0, 1$

say : $k=3$, define $\tilde{r} = \sqrt{3}\,r$, $\tilde{\mathcal{R}} = \frac{1}{\sqrt{3}}\mathcal{R}$

$$\Rightarrow dl^2 = \tilde{\mathcal{R}}^2\left(\frac{d\tilde{r}^2}{1-\tilde{r}^2} + \tilde{r}^2 d\Omega^2\right)$$

⇒ THREE TYPES OF UNIVERSES

FLAT UNIVERSE:

$k=0$, $d\ell^2 = R^2 dr^2 + R^2 r^2 d\Omega^2$

Setting $\tilde{r} = Rr$ it is easy to recognize the metric of a flat Euclidean space (expanding!)

CLOSED UNIVERSE:

$$k=1, \quad d\ell^2 = R^2 \frac{dr^2}{1-r^2} + R^2 r^2 d\Omega^2$$

We can define $\chi(r)$ such that: $d\chi^2 = \frac{dr^2}{1-r^2}$

this is easily integrated $(\chi(r=0)=0)$:

$$r = \sin\chi \qquad 0 \le \chi \le \pi$$

$$\Rightarrow d\ell^2 = R^2\left[d\chi^2 + \sin^2\chi\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)\right]$$

It can be demonstrated that this is the line element of a 3-sphere of radius R, embedded in a flat 4D space

BUT: this is of no relevance, a curved space DOES NOT NEED to be a hypersurface of a higher dimensional flat space

OPEN UNIVERSE:

$$k=-1, \quad d\ell^2 = R^2 \frac{dr^2}{1+r^2} + R^2 r^2 d\Omega^2$$

We can define $\chi(r)$ such that: $d\chi^2 = \frac{dr^2}{1+r^2}$

this is easily integrated $(\chi(r=0)=0)$:

$$r = \sinh\chi \qquad \chi \ge 0$$

$$\Rightarrow d\ell^2 = R^2\left[d\chi^2 + \sinh^2\chi\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)\right]$$

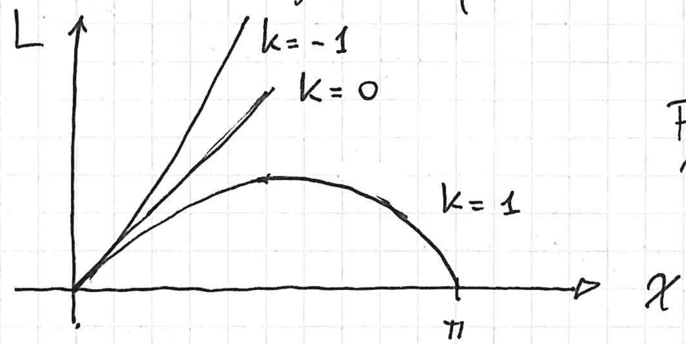
This is NOT a hypersurface of a flat 4D space

Summing up, we can define a χ coordinate:

$$r = \begin{cases} \sin\chi & k=1 \quad \text{closed} \\ \chi & k=0 \quad \text{flat} \\ \sinh\chi & k=-1 \quad \text{open} \end{cases}$$

Length of a great circle: $r = \text{const}$, $\vartheta = \frac{\pi}{2}$

$$dl^2 = R^2 r^2 d\varphi^2 \Rightarrow L(\chi) = 2\pi R r = \begin{cases} 2\pi R \sin\chi & k=1 \\ 2\pi R\, \chi & k=0 \\ 2\pi R \sinh\chi & k=-1 \end{cases}$$



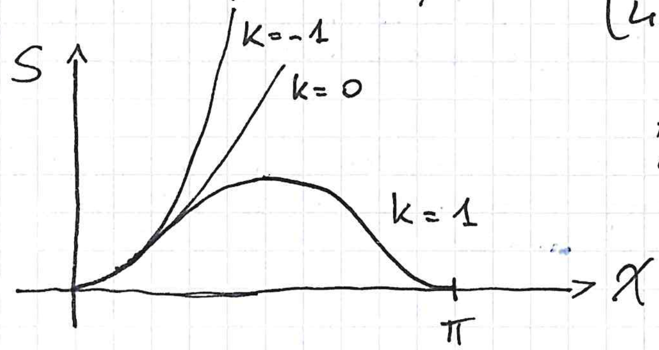
For $k=1$ $L(\chi)$ reaches a maximum and then closes, like a 2-sphere

Radius of this great circle: $\vartheta = \varphi = \text{const}$, $dl^2 = R^2 d\chi^2$

$$\Rightarrow R\chi$$

Area of a surface at $r = \text{const}$: $dl^2 = R^2 r^2 d\Omega^2$

$$\Rightarrow S(\chi) = 4\pi R^2 r^2 = \begin{cases} 4\pi R^2 \sin^2\chi & k=1 \\ 4\pi R^2 \chi^2 & k=0 \\ 4\pi R^2 \sinh^2\chi & k=-1 \end{cases}$$

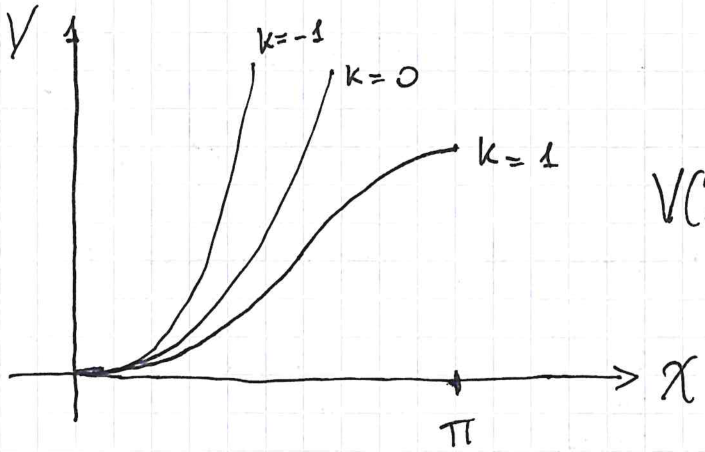


Again, for $k=1$ the area reache's a maximum and then decreases

Volume within a coordinate r:

We must integrate $V(\chi) = \int \sqrt{g}\, d\chi\, d\vartheta\, d\varphi = \int dV$

$$dV = \frac{R^3 r^2 \sin\vartheta}{\sqrt{1 - kr^2}}\, dr\, d\vartheta\, d\varphi = R^3 \sin\vartheta\, d\chi\, d\vartheta\, d\varphi \times \begin{cases} \sin^2\chi \\ \chi^2 \\ \sinh^2\chi \end{cases}$$



$$V(\chi) = \begin{cases} 2\pi R^3 \left(\chi - \frac{1}{2}\sin 2\chi\right) & k=1 \\ \frac{4}{3}\pi R^3 \chi^3 & k=0 \\ 2\pi R^3 \left(\frac{1}{2}\sinh 2\chi - \chi\right) & k=-1 \end{cases}$$

→ Defining $R_0 \equiv R(t=t_0)$, where t_0 is the present age, we can define the ADIMENSIONAL SCALE FACTOR:

$$a(t) \equiv \frac{R(t)}{R(t_0)} \quad , \quad a(t=t_0) = 1$$

Calling $\tilde{r} = R_0 r$, we can write the FRW metrices:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{d\tilde{r}^2}{1-\frac{k}{R_0^2}\tilde{r}^2} + \tilde{r}^2 d\Omega^2\right]$$

In the following we will use this expression with r in place of $\tilde{r}$

One can write this as follows:

Line-of-sight distance $dL_{los} = R_0\, d\chi$

Distance on the sky plane $L_{sky} = R_0 r = R_0 \begin{cases} \sin\chi & k=1 \\ \chi & k=0 \\ \sinh\chi & k=-1 \end{cases}$

$$ds^2 = -dt^2 + a^2(t)\left[dL_{los}^2 + L_{sky}^2\, d\Omega^2\right]$$

Note: L_{los} and L_{sky} are COMOVING DISTANCES

→ One can also define the CONFORMAL TIME:

$$d\eta = \frac{dt}{a(t)}$$

$$ds^2 = a^2(t)\left[-d\eta^2 + dL_{los}^2 + L_{sky}^2\, d\Omega^2\right]$$