Lecture 7 — The Hubble law

Schutz, Cap 12
Vittorio', Caps 1-2

COSMOLOGICAL REDSHIFT

NOTE: every problem involving light propagation implies
(1) something that emits a photon
(2) the photon propagating on a null geodesic
(3) the observer that receives the photon

In a FRW massive objects corresponding to emitters and receivers ("galaxies" in both cases) travel along time-like geodesics and have constant positions in comoving coordinates

$\Rightarrow$ L_{LOS} is constant

Let's have a photon propagating along the line of sight.

WE EXPLICIT c HERE

$$ds^2 = -c^2 dt^2 + a^2 \, dL_{LOS}^2 = 0 \Rightarrow \frac{c\,dt}{a(t)} = dL_{LOS}$$

The emitting galaxies gives two photons at t_{em} and $t_{em} + \Delta t_{em}$, observed at t_{obs} and Δt_{obs}

$$\int_{t_{em}}^{t_{obs}} \frac{c\,dt}{a(t)} = \int_{GAL\,em}^{GAL\,obs} dL_{LOS} = \int_{t_{em}+\Delta t_{em}}^{t_{obs}+\Delta t_{obs}} \frac{c\,dt}{a(t)}$$

This implies that:

$$\int_{t_{em}}^{t_{em}+\Delta t_{em}} \frac{c\,dt}{a(t)} \simeq \frac{c\,\Delta t_{em}}{a(t_{em})} = \int_{t_{obs}}^{t_{obs}+\Delta t_{obs}} \frac{c\,dt}{a(t)} \simeq c\,\Delta t_{obs}$$

given that $a(t_{obs}) = 1$. Given that frequencies are $\propto t^{-1}$ and $\lambda \propto \nu^{-1}$

$$\Rightarrow \frac{1}{a(t_{em})} = \frac{\lambda_{obs}}{\lambda_{em}} = 1 + \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}} = 1 + z_{em}$$

The Schutz textbook gives a proof that is similar to that of gravitational redshift:

the metric $ds^2 = -c^2 dt^2 + a^2(t) \, R_0^2 \, d\chi^2$ does not depend on χ $\Rightarrow$ p_χ is conserved.

Using $p^\alpha p_\alpha = 0$ we get:

$$0 = -p_0^2 + \frac{1}{R_0^2 a^2(t)} p_\chi^2 \quad \Rightarrow \quad p_0 \propto \frac{1}{a(t)}$$

The photon energy in the frame of the observer, for which

$$\vec{U} = (1,0,0,0)$$

is: $\quad E = -\vec{U}\cdot\vec{p} = -p_0 = h\nu_{obs} \quad \Rightarrow \quad h\nu_{obs} \propto \frac{1}{a(t)} \Rightarrow \lambda_{obs} \propto a$

$$\Rightarrow \quad \frac{\lambda_{obs}}{\lambda_{em}} = \frac{1}{a(t_{em})} \qquad \text{again}$$

But the basic interpretation is: λ expands with the Universe, as all lengths do.

PROPER DISTANCE

Keep $t = \text{const}$, $a(t) = \text{const}$, the distance between two galaxies is:

$$d_p = \int_0^r a(t) \frac{dr'}{\sqrt{1 - \frac{k}{R_0^2} r'^2}} = a(t)\, f(r) = a(t) R_0 \chi(r)$$

$$= a(t)\, L_{los}$$

(different definitions in different textbooks)

This is of course NOT measurable

COMOVING DISTANCE

already discussed

$$d_c = f(r) = R_0 \chi(r) = L_{los} \quad , \quad d_p = a(t)\, d_c$$

HUBBLE PARAMETER

Let's call "velocity" v the rate at which the proper distance changes with time:

$$v = \frac{d}{dt} d_p = \dot{a}(t)\, d_c = \frac{\dot{a}}{a} d_p$$

where the dot denotes time derivative.

Let's define the Hubble parameter as:

$$H(t) = \frac{\dot{a}}{a}$$

then: $\quad v = H d_p$

but THIS IS NOT THE HUBBLE LAW in a strict sense

because d_p is not measurable

To understand observable distances we adopt an OPERATIONAL APPROACH

We measure distances through:

- standard candles
- standard rulers

→ Standard candle: $f = \frac{L}{4\pi d^2} = \frac{L}{S}$

f is the measured flux (T^{0i})
L the candle luminosity
S the area of a surface at $r = d$

$S = 4\pi R_0^2 r^2$ where r is the coordinate distance

Here we use R_0 because the photons are observed at $t = t_0$

T^{0i} is affected twice by a transformation:

L is energy / time, $E = h\nu \propto \frac{1}{a}$, $dt \propto a$

$\Rightarrow L \propto a^{-2}$, $L_{obs} = L_{em}\, a^2$

$\Rightarrow f = \frac{L_{em}}{4\pi R_0^2 r^2} a_e^2 = \frac{L_{em}}{4\pi d_L^2}$

LUMINOSITY DISTANCE

$$d_L = \frac{1}{a} R_0 r = (1+z) R_0 r = (1+z) L_{SKY}$$

→ Standard ruler: $\vartheta = \frac{D}{d}$ (in radians)

ϑ is the angular exension of an object
D is its physical size

Expansion is isotropic and keeps the angles constant, so we can compute this at fixed time

We use the time at emission, because D is defined at that time.

We use the computation of a great circle to obtain:

$$D = \vartheta\, a R_0 r \Rightarrow \vartheta = \frac{D}{a R_0 r} = \frac{D}{d_D}$$

DIAMETER DISTANCE

$$d_D = a R_0 r = \frac{1}{1+z} R_0 r = \frac{1}{1+z} L_{SKY}$$

It follows that: $d_D (1+z)^2 = d_L$

The relation between d_L and d_C is:

$$d_L = \begin{cases} (1+z) R_0 \sin\left(\frac{d_C}{R_0}\right) & k=1 \\ (1+z)\, d_C & k=0 \\ (1+z) R_0 \sinh\left(\frac{d_C}{R_0}\right) & k=-1 \end{cases}$$

and similarly for d_D, with $(1+z)^{-1}$ in place of $(1+z)$

NB: a peculiarity of diameter distance is that it does not necessarily increase with z or d_C

COSMOLOGICAL DIMMING

The surface brightness of an object is its luminosity per unit area:

$$I = \frac{L}{D^2} \quad , \text{ if } L \propto \frac{1}{r} \text{ and } D \propto \frac{1}{r} \text{ then } I = \text{const}$$

but in cosmology: $I = I_0 \left(\frac{d_D}{d_L}\right)^2 = I_0 (1+z)^{-4}$

HUBBLE LAW

Let's Taylor-expand $a(t)$ to first order:

$$a(t) = a(t_0) + (t-t_0)\dot{a}(t_0) + O(|t-t_0|^2)$$

Calling $H_0 = \frac{\dot{a}(t_0)}{a(t_0)} = \dot{a}(t_0) \qquad (a(t_0)=1)$

$$\frac{1}{1+z} = a(t) \simeq 1 + H_0(t-t_0) + O(|t-t_0|^2)$$

$$1+z \simeq 1 - H_0(t-t_0) \implies z \simeq H_0(t_0-t)$$

Photons travel along null geodesics:

$$ds^2 = -c^2 dt^2 + a^2(t) \frac{R_0^2 dr^2}{1-kr^2} = 0$$

$$\int_t^{t_0} \frac{c\,dt}{a(t)} = \int_0^r \frac{R_0 dr'}{\sqrt{1-kr'^2}}$$

The right-hand side is expanded as:

$$\int_0^r \frac{R_0 dr'}{\sqrt{1-kr'^2}} = R_0 r + \mathcal{O}(|R_0 r|^3)$$

while the left-hand side is:

$$\int_t^{t_0} \frac{c\,dt}{a(t)} = c(t_0 - t) + \mathcal{O}(|t - t_0|^2) \simeq \frac{cz}{H_0}$$

To the same first order: $d_L \simeq d_D \simeq d_p \simeq d_c \simeq R_0 r$

$$\Rightarrow \quad cz = H_0 d_L$$

This shows that:

+ z IS NOT a Doppler redshift
+ $v = cz$ IS NOT a velocity in the usual sense
+ z is a gravitational redshift due to time dilation in an expanding Universe

SUPERLUMINAL MOTIONS

Consider the "Hubble law" in the form:

$$v = H d_p$$

this is how unobservable proper distance varies with time.

It is possible that: $v = H d_p > c$

This is NOT a violation of SR, that applies to objects passing through our (local) inertial frame, so that their velocity can be measured

DECELERATION PARAMETER

The relations above can be expanded to second order

$$a(t) \simeq 1 + H_0(t - t_0) - \frac{1}{2} q_0 H_0^2 (t - t_0)^2 + \dots$$

$$q_0 \equiv - \frac{\ddot{a}(t_0)}{\dot{a}(t_0)^2} \qquad \text{Deceleration parameter}$$

$$z \simeq H_0(t_0 - t) + \left(1 + \frac{1}{2} q_0\right) H_0^2 (t - t_0)^2 + \dots$$

$$\int_t^{t_0} \frac{c\,dt}{a(t)} \simeq c\left[(t_0-t)+\frac{1}{2}H_0(t_0-t)^2+\dots\right]$$

$$\simeq \frac{c}{H_0}\left[z-\frac{1}{2}(1+q_0)z^2+\dots\right]$$

$$d_L=(1+z)R_0 r \simeq \frac{c}{H_0}\left[z+\frac{1}{2}(1-q_0)z^2+\dots\right]$$

This last, observable relation makes it possible to measure q_0 from the non linearity of Hubble law at $z \gtrsim 1$

We will see that this parameter is related to the matter/energy content of the universe, that is responsible for its deceleration - or acceleration.