Lecture 8

Friedmann equations from Einstein equations

Schutz, Cap 12; Carroll; Vittorio, Cap. 1

We want to work out the evolution of $a(t)$, that will depend on what the universe contains

We start from the FRW metric written as:

$$ds^2 = -dt^2 + a^2\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]$$

NB: a is dimensionless, r is a length, k should be k/R_0^2 but we keep the notation light

$$g_{\alpha\beta} = \mathrm{diag}\left(-1, \frac{a^2}{1-kr^2}, a^2r^2, a^2r^2\sin^2\vartheta\right)$$

$$g^{\alpha\beta} = \mathrm{diag}\left(-1, \frac{1-kr^2}{a^2}, \frac{1}{a^2r^2}, \frac{1}{a^2r^2\sin^2\vartheta}\right)$$

$$g_{\alpha\beta,0} = \mathrm{diag}\left(0, \frac{2a\dot{a}}{1-kr^2}, 2a\dot{a}r^2, 2a\dot{a}r^2\sin^2\vartheta\right)$$

$$g_{\alpha\beta,1} = \mathrm{diag}\left(0, \frac{2a^2kr}{(1-kr^2)^2}, 2a^2r, 2a^2r\sin^2\vartheta\right)$$

$$g_{\alpha\beta,2} = \mathrm{diag}\left(0,0,0,2a^2r^2\sin\vartheta\cos\vartheta\right), \quad g_{\alpha\beta,3} = 0$$

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\left(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu}\right)$$

$\mu = 0$:

$$\Gamma^0{}_{\alpha\beta} = \frac{1}{2}g^{00}\left(g_{0\alpha,\beta} + g_{0\beta,\alpha} - g_{\alpha\beta,0}\right)$$

non-zero for $\alpha = \beta$ except $\alpha = \beta = 0$

$$\Gamma^0{}_{00} = \frac{1}{2}g^{00}g_{00,0} = 0$$

$$\Gamma^0{}_{11} = -\frac{1}{2}g^{00}g_{11,0} = \frac{a\dot{a}}{1-kr^2}$$

$$\Gamma^0{}_{22} = -\frac{1}{2}g^{00}g_{22,0} = a\dot{a}r^2$$

$$\Gamma^0{}_{33} = -\frac{1}{2}g^{00}g_{33,0} = a\dot{a}r^2\sin^2\vartheta$$

$\mu = 1$:

$$\Gamma^1_{\alpha\beta} = \frac{1}{2} g^{11}\left(g_{1\alpha,\beta} + g_{1\beta,\alpha} - g_{\alpha\beta,1}\right)$$

non-zero:

$\alpha=\beta\neq 0$,

$\alpha=1$, $\beta=0$ & $\alpha=0, \beta=1$

$$\Gamma^1_{11} = \frac{1}{2} g^{11} g_{11,1} = \frac{kr}{1-kr^2}$$

$$\Gamma^1_{22} = -\frac{1}{2} g^{11} g_{22,1} = -r(1-kr^2)$$

$$\Gamma^1_{33} = -\frac{1}{2} g^{11} g_{33,1} = -r(1-kr^2)\,\mathrm{sen}^2\vartheta$$

$$\Gamma^1_{10} = \Gamma^1_{01} = \frac{1}{2} g^{11} g_{11,0} = \frac{\dot{a}}{a}$$

$\mu = 2$

$$\Gamma^2_{\alpha\beta} = \frac{1}{2} g^{22}\left(g_{2\alpha,\beta} + g_{2\beta,\alpha} - g_{\alpha\beta,2}\right)$$

non-zero:

$\alpha=3$, $\beta=3$

$\alpha=2$, $\beta=0$ & vv

$\alpha=2$, $\beta=1$ & vv

$$\Gamma^2_{33} = -\frac{1}{2} g^{22} g_{33,2} = -\mathrm{sen}\,\vartheta\cos\vartheta$$

$$\Gamma^2_{20} = \Gamma^2_{02} = \frac{1}{2} g^{22} g_{22,0} = \frac{\dot{a}}{a}$$

$$\Gamma^2_{21} = \Gamma^2_{12} = \frac{1}{2} g^{22} g_{22,1} = \frac{1}{r}$$

$\mu = 3$

$$\Gamma^3_{\alpha\beta} = \frac{1}{2} g^{33}\left(g_{3\alpha,\beta} + g_{3\beta,\alpha} - g_{\alpha\beta,3}\right)$$

nonzero:

$\alpha=3$, $\beta=0,1,2$

$$\Gamma^3_{30} = \Gamma^3_{03} = \frac{1}{2} g^{33} g_{33,0} = \frac{\dot{a}}{a}$$

$$\Gamma^3_{31} = \Gamma^3_{13} = \frac{1}{2} g^{33} g_{33,1} = \frac{1}{r}$$

$$\Gamma^3_{32} = \Gamma^3_{23} = \frac{\cos\vartheta}{\mathrm{sen}\,\vartheta}$$

Ricci tensor:

$$R^\mu{}_{\alpha\mu\beta} = \Gamma^\mu_{\alpha\beta,\mu} - \Gamma^\mu_{\alpha\mu,\beta} + \Gamma^\mu_{\sigma\mu}\Gamma^\sigma_{\alpha\beta} - \Gamma^\mu_{\sigma\beta}\Gamma^\sigma_{\alpha\mu}$$

It is useful to compute the term $\Gamma^{\mu}{}_{\alpha\mu}$:

$$\Gamma^{\mu}{}_{0\mu} = \cancel{\Gamma^{0}{}_{00}} + \Gamma^{1}{}_{01} + \Gamma^{2}{}_{02} + \Gamma^{3}{}_{03} = 3\,\frac{\dot a}{a}$$

$$\Gamma^{\mu}{}_{1\mu} = \cancel{\Gamma^{0}{}_{10}} + \Gamma^{1}{}_{11} + \Gamma^{2}{}_{12} + \Gamma^{3}{}_{13} = \frac{kr}{1-kr^2} + \frac{2}{r}$$

$$\Gamma^{\mu}{}_{2\mu} = \cancel{\Gamma^{0}{}_{20}} + \cancel{\Gamma^{1}{}_{21}} + \cancel{\Gamma^{2}{}_{22}} + \Gamma^{3}{}_{23} = \frac{\cos\vartheta}{\operatorname{sen}\vartheta}$$

$$\Gamma^{\mu}{}_{3\mu} = \cancel{\Gamma^{0}{}_{30}} + \cancel{\Gamma^{1}{}_{31}} + \cancel{\Gamma^{2}{}_{32}} + \cancel{\Gamma^{3}{}_{33}} = 0$$

For the other terms, we can anticipate that $R_{\alpha\beta}$ is diagonal

We compute here $\Gamma^{\mu}{}_{\sigma\alpha}\Gamma^{\sigma}{}_{\mu\beta}$ for $\alpha=\beta$:

NB: Γ not reported symmetry exploited

00

$$\cancel{{}^{0}_{00}\,{}^{0}_{00}} + \cancel{{}^{1}_{00}\,{}^{0}_{10}} + \cancel{{}^{2}_{00}\,{}^{0}_{20}} + \cancel{{}^{3}_{00}\,{}^{0}_{30}} +$$

$$\| \quad + {}^{1}_{10}\,{}^{1}_{10} + \cancel{{}^{2}_{10}\,{}^{1}_{20}} + \cancel{{}^{3}_{10}\,{}^{1}_{30}} +$$

Symmetric terms

$$+ {}^{2}_{20}\,{}^{2}_{20} + \cancel{{}^{3}_{20}\,{}^{2}_{30}} +$$

$$+ {}^{3}_{30}\,{}^{3}_{30} = \left(\Gamma^{1}{}_{10}\right)^2 + \left(\Gamma^{2}{}_{20}\right)^2 + \left(\Gamma^{3}{}_{30}\right)^2 = 3\left(\frac{\dot a}{a}\right)^2$$

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$$\cancel{{}^{0}_{01}\,{}^{0}_{01}} + {}^{1}_{01}\,{}^{0}_{11} + \cancel{{}^{2}_{01}\,{}^{0}_{21}} + \cancel{{}^{3}_{01}\,{}^{0}_{31}} +$$

$$+ {}^{1}_{11}\,{}^{1}_{11} + \cancel{{}^{2}_{11}\,{}^{1}_{21}} + \cancel{{}^{3}_{11}\,{}^{1}_{31}} +$$

$\frac{\dot a^2}{1-kr^2}$ $\quad$ $\left(\frac{kr}{1-kr^2}\right)^2$

$$+ {}^{2}_{21}\,{}^{2}_{21} + \cancel{{}^{3}_{21}\,{}^{2}_{31}} +$$

$\frac{1}{r^2}$

$$+ {}^{3}_{31}\,{}^{3}_{31} = \frac{2\dot a^2}{1-kr^2} + \left(\frac{kr^2}{1-kr^2}\right)^2 + \frac{2}{r^2}$$

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$$\cancel{{}^{0}_{02}\,{}^{0}_{02}} + \cancel{{}^{1}_{02}\,{}^{0}_{12}} + {}^{2}_{02}\,{}^{0}_{22} + \cancel{{}^{3}_{02}\,{}^{0}_{32}} +$$

$$+ \cancel{{}^{1}_{12}\,{}^{1}_{12}} + {}^{2}_{12}\,{}^{1}_{22} + \cancel{{}^{3}_{12}\,{}^{1}_{32}} +$$

$\dot a^2 r^2$

$-(1-kr^2)$

$$+ \cancel{{}^{2}_{22}\,{}^{2}_{22}} + \cancel{{}^{3}_{22}\,{}^{2}_{32}} +$$

$\left(\frac{\cos\vartheta}{\operatorname{sen}\vartheta}\right)^2$

$$+ {}^{3}_{32}\,{}^{3}_{32} = 2\dot a^2 r^2 - 2(1-kr^2) + \left(\frac{\cos\vartheta}{\operatorname{sen}\vartheta}\right)^2$$

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$$\cancel{{}^{0\ 0}_{0303}} + \cancel{{}^{1\ 1}_{0313}} + \cancel{{}^{2\ 0}_{0323}} + {}^{3\ 0}_{0333} + \quad \rightsquigarrow \dot a^2 r^2 \,\mathrm{sen}^2\vartheta$$

$$+ \cancel{{}^{1\ 1}_{1313}} + \cancel{{}^{2\ 1}_{1323}} + {}^{3\ 1}_{1333} + \quad \rightsquigarrow -(1-kr^2)\,\mathrm{sen}^2\vartheta$$

$$+ \cancel{{}^{2\ 2}_{2323}} + {}^{3\ 2}_{2333} + \quad \rightsquigarrow -\cos^2\vartheta$$

$$+ \cancel{{}^{3\ 3}_{3333}} = 2\dot a^2 r^2 \,\mathrm{sen}^2\vartheta - 2(1-kr^2)\,\mathrm{sen}^2\vartheta - 2\cos^2\vartheta$$

R_{00} :

$$R_{00} = \cancel{\Gamma^{\mu}{}_{00,\mu}} - \Gamma^{\mu}{}_{0\mu,0} + \cancel{\Gamma^{\mu}{}_{\sigma\mu}\Gamma^{\sigma}{}_{00}} - \Gamma^{\mu}{}_{\sigma 0}\Gamma^{\sigma}{}_{\mu 0} =$$

$$= -\frac{d}{dt}\left(3\frac{\dot a}{a}\right) - 3\left(\frac{\dot a}{a}\right)^2 = -3\frac{\ddot a}{a}$$

$$R_{11} = \Gamma^{\mu}{}_{11,\mu} - \Gamma^{\mu}{}_{1\mu,1} + \Gamma^{\mu}{}_{\sigma\mu}\Gamma^{\sigma}{}_{11} - \Gamma^{\mu}{}_{\sigma 1}\Gamma^{\sigma}{}_{\mu 1} =$$

$$= \frac{d}{dt}\left(\frac{a\dot a}{1-kr^2}\right) + \cancel{\frac{d}{dr}\left(\frac{kr}{1-kr^2}\right)} - \frac{d}{dr}\left(\cancel{\frac{kr}{1-kr^2}} + \frac{2}{r}\right) + \frac{3\dot a}{a}\frac{a\dot a}{1-kr^2} +$$

$$+ \left(\cancel{\frac{kr}{1-kr^2}} + \frac{2}{r}\right)\frac{kr}{1-kr^2} - \frac{2\dot a^2}{1-kr^2} - \cancel{\left(\frac{kr}{1-kr^2}\right)^2} - \frac{2}{r^2} =$$

$$= \frac{a\ddot a + 2\dot a^2 + 2k}{1-kr^2}$$

$$R_{22} = \Gamma^{\mu}{}_{22,\mu} - \Gamma^{\mu}{}_{2\mu,2} + \Gamma^{\mu}{}_{\sigma\mu}\Gamma^{\sigma}{}_{22} - \Gamma^{\mu}{}_{\sigma 2}\Gamma^{\sigma}{}_{\mu 2} =$$

$$= \frac{d}{dt}(a\dot a r^2) - \frac{d}{dr}\left[r(1-kr^2)\right] - \frac{d}{d\vartheta}\frac{\cos\vartheta}{\mathrm{sen}\vartheta} + 3\frac{\dot a}{a}a\dot a r^2 -$$

$$- \left(\frac{kr}{1-kr^2} + \cancel{\frac{2}{r}}\right) r(1-kr^2) - 2\dot a^2 r^2 + \cancel{2(1-kr^2)} - \left(\frac{\cos\vartheta}{\mathrm{sen}\vartheta}\right) =$$

$$= r^2\left(a\ddot a + 2\dot a^2 + 2k\right)$$

$$R_{33} = \Gamma^{\mu}{}_{33,\mu} - \Gamma^{\mu}{}_{3\mu,3} + \Gamma^{\mu}{}_{\sigma\mu}\Gamma^{\sigma}{}_{33} - \Gamma^{\mu}{}_{\sigma 3}\Gamma^{\sigma}{}_{\mu 3} =$$

$$= \frac{d}{dt}(a\dot a r^2 \,\mathrm{sen}^2\vartheta) - \frac{d}{dr}\left[r(1-kr^2)\,\mathrm{sen}^2\vartheta\right] - \frac{d}{d\vartheta}\,\mathrm{sen}\vartheta\cos\vartheta +$$

$$+ 3\frac{\dot a}{a} a\dot a r^2 \operatorname{sen}^2\vartheta - \left(\frac{kr}{1-kr^2} + \frac{2}{r}\right) r(1-kr^2)\operatorname{sen}^2\vartheta +$$

$$- \frac{\cos\vartheta}{\operatorname{sen}\vartheta} \operatorname{sen}\vartheta\cos\vartheta - 2\dot a^2 r^2 \operatorname{sen}^2\vartheta + 2(1-kr^2)\operatorname{sen}^2\vartheta + 2\cos^2\vartheta =$$

$$= r^2(a\ddot a + 2\dot a^2 + 2k)\operatorname{sen}^2\vartheta$$

It is easy to see that the space part of $R_{\alpha\beta}$ is:

$$R_{ij} = (a\ddot a + 2\dot a^2 + 2k)\,\mathrm{diag}\left(\frac{1}{1-kr^2}, r^2, r^2\operatorname{sen}^2\vartheta\right)$$

$$= \frac{1}{a^2}(a\ddot a + 2\dot a^2 + 2k)\, g_{ij} = \left(\frac{\ddot a}{a} + 2\frac{\dot a^2}{a^2} + \frac{2k}{a^2}\right) g_{ij}$$

Ricci scalar:

$$R = g^{\alpha\beta}R_{\alpha\beta} = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} =$$

$$= -R_{00} + \left(\frac{\ddot a}{a} + 2\frac{\dot a^2}{a^2} + 2\frac{k}{a^2}\right) g^{ij}g_{ij} =$$

$$= 3\frac{\ddot a}{a} + 3\left(\frac{\ddot a}{a} + 2\frac{\dot a^2}{a^2} + 2\frac{k}{a^2}\right) = 6\left(\frac{\ddot a}{a} + \frac{\dot a^2}{a^2} + \frac{k}{a^2}\right)$$

It is convienent to solve the Einstein equations in the form:

$$R_{\alpha\beta} = 8\pi G\left(T_{\alpha\beta} - \frac{1}{2}T g_{\alpha\beta}\right)$$

We assume a perfect fluid:

$$T^{\alpha\beta} = (P+\rho)U^\alpha U^\beta + P g^{\alpha\beta} = \mathrm{diag}(\rho, P g^{ij})$$

$$T = g_{\alpha\beta}T^{\alpha\beta} = (P+\rho)\, g_{\alpha\beta}U^\alpha U^\beta + g_{\alpha\beta}g^{\alpha\beta}P =$$

$$= -(P+\rho) + 4P = -\rho + 3P$$

(NB: because $g_{00} = -1$, $U^0 = \sqrt{-\frac{1}{g_{00}}} = 1$, $U_0 = -1$)

00 component of the equation:

$$R_{00} = 8\pi G\left(T_{00} - \tfrac{1}{2} T g_{00}\right)$$

$$-3\frac{\ddot{a}}{a} = 8\pi G\left[\rho + \tfrac{1}{2}(-\rho + 3P)\right] = 4\pi G(\rho + 3P)$$

(1) $$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)}$$

ij component of the equation:

$$R_{ij} = 8\pi G\left(T_{ij} - \tfrac{1}{2} T g_{ij}\right) \qquad \text{all} \propto g_{ij}$$

$$\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{k}{a^2} = 8\pi G\left[P - \tfrac{1}{2}(-\rho + 3P)\right] = 4\pi G(\rho - P)$$

$$-\frac{4\pi G}{3}(\rho + \cancel{3P}) + 2\frac{\dot{a}^2}{a^2} + 2\frac{k}{a^2} = 4\pi G(\rho - \cancel{P})$$

(2) $$\boxed{\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{k}{a^2}}$$

The same equation can be obtained from the trace of the Einstein equation:

$$R = 8\pi G(T - 2T) = -8\pi G T$$

$$6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) = -8\pi G(-\rho + 3P)$$

$$-\frac{4\pi G}{3}(\rho + \cancel{3P}) + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -\frac{4\pi G}{3}(-\rho + \cancel{3P})$$

$$\Rightarrow \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

Energy conservation takes this form (see also Vittorio, sez. 1.6):

$$T^{\alpha}{}_{\beta} = (\rho + P)U^{\alpha}U_{\beta} + P\,\delta^{\alpha}{}_{\beta} = \mathrm{diag}(-\rho, P, P, P)$$

$$T^{\mu}{}_{\nu;\mu} = T^{\mu}{}_{\nu,\mu} + \Gamma^{\mu}{}_{\sigma\mu} T^{\sigma}{}_{\nu} - \Gamma^{\sigma}{}_{\nu\mu} T^{\mu}{}_{\sigma} = 0$$

all space derivatives vanish, for $\nu = 0$:

$$T^{0}{}_{0,0} + \Gamma^{\mu}{}_{0\mu} T^{0}{}_{0} - \Gamma^{1}{}_{01} T^{1}{}_{1} - \Gamma^{2}{}_{02} T^{2}{}_{2} - \Gamma^{3}{}_{03} T^{3}{}_{3} =$$

$$= -\frac{d}{dt}\rho - 3\frac{\dot{a}}{a}\rho - 3\frac{\dot{a}}{a}P = 0$$

③ $$\boxed{\dot\rho = -3\frac{\dot a}{a}(P+\rho)}$$

This is how density changes under Hubble expansion, and it contains pressure

These three equations are not independent: take the time derivative of the second

$$\frac{d}{dt}\left(\frac{\dot a^2}{a^2}\right) = 2\frac{\dot a}{a}\left(\frac{\ddot a}{a} - \frac{\dot a^2}{a^2}\right) = \frac{8\pi G}{3}\dot\rho + 2k\frac{\dot a}{a^3}$$

then use the third equation to obtain $\dot\rho$ and the second again for $\dot a^2/a^2$:

$$2\frac{\dot a}{a}\left(\frac{\ddot a}{a} - \frac{8\pi G}{3}\rho + \cancel{\frac{k}{a^2}}\right) = -8\pi G\frac{\dot a}{a}(P+\rho) + \cancel{2k\frac{\dot a}{a^3}}$$

$$\Rightarrow \frac{\ddot a}{a} = \frac{8\pi G}{3}\rho - 4\pi G(P+\rho) = -\frac{4\pi G}{3}(P+3\rho)$$

We can write the third equation as follows:

$$a^3 d\rho = -3a^2 da(P+\rho)$$

$$a^3 d\rho + 3a^2\rho\, da + 3a^2 da\, P = 0$$

$$d(\rho a^3) + P\, d(a^3) = 0$$

Calling $a^3 = V$, $U = \rho a^3$ $\Rightarrow$ $dU + P dV = 0$

so this is the first law of thermodynamics.

CRITICAL DENSITY

Consider the flat case $k=0$:

$$\frac{\dot a^2}{a^2} = H^2(t) = \frac{8\pi G}{3}\rho(t) \Rightarrow \rho = \frac{3H^2}{8\pi G} \equiv \rho_c(t)$$

then: $$\rho_{c0} = \rho_c(t_0) = \frac{3H_0^2}{8\pi G}$$

DENSITY PARAMETER

$$\Omega(t) = \frac{\rho(t)}{\rho_c(t)}, \quad \Omega_0 = \frac{\rho_0}{\rho_{c0}}$$

The second Friedmann equation becomes:

$$\Omega(t) = 1 + \frac{k}{H^2 a^2}$$

so if $k = 1$ $\quad \Omega(t) > 1 \quad , \quad \rho > \rho_c$

if $k = -1$ $\quad \Omega(t) < 1 \quad , \quad \rho < \rho_c$

Thus the critical density seperates closed and open universes.

$$\Rightarrow \quad k = (\Omega - 1) H^2 a^2 = (\Omega_0 - 1) H_0^2$$

and $\quad \Omega_0 = \dfrac{\rho_0}{\rho_{c0}} = \dfrac{\rho_0}{H_0^2} \dfrac{8\pi G}{3} \quad \Rightarrow \quad \dfrac{8\pi G}{3} = \dfrac{\Omega_0 H_0^2}{\rho_0}$

so the second Friedmann equation becomes:

$$H^2 = \frac{\Omega_0 H_0^2}{\rho_0} \rho - \frac{1}{a^2} (\Omega_0 - 1) H_0^2 \qquad \text{or}$$

$$E^2(t) \equiv \frac{H^2}{H_0^2} = \Omega_0 \frac{\rho}{\rho_0} + (1 - \Omega_0) a^{-2}$$

BACK TO C

If we use a system of units for which $c \neq 1$:

$$ds^2 = -c^2 dt^2 + a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right)$$

going through the calculations, one gets:

$$R = \frac{6}{c^2} \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right)$$

and $\quad T_{\alpha\beta} = \mathrm{diag}(\rho c^2, P g_{ij}) \quad , \quad T = -\rho c^2 + 3P$

$$R = -\frac{8\pi G}{c^4} T \qquad \text{gives:}$$

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3} \rho - \frac{kc^2}{a^2} \quad , \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left(\rho + 3 \frac{P}{c^2} \right)$$

BACK TO R_0^2

$$\dot{\rho} = -3 \frac{\dot{a}}{a} \left(\frac{P}{c^2} + \rho \right)$$

The second Friedmann equation should read:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3} \rho - \frac{kc^2}{R_0^2} \frac{1}{a^2} \quad , \quad \frac{kc^2}{R_0^2} = (\Omega - 1) H^2 a^2$$