Lecture 9

The Einstein-de Sitter model

Vittorio, chaps. 1 and 2

The simplest case is that of a flat, matter-dominated universe

$k=0, \quad p=0$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho \quad \Rightarrow \quad \text{the universe decelerates}$$

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho \quad \Rightarrow \quad \rho = \rho_c = \frac{3H^2}{8\pi G}$$

$$\dot{\rho} = -3\frac{\dot{a}}{a}\rho \quad \Rightarrow \quad \rho = \rho_{c0}\, a^{-3}, \quad \rho_{c0} = \frac{3H_0^2}{8\pi G}$$

$$\Rightarrow H^2 = H_0^2 a^{-3}$$

$$\Rightarrow \dot{a}^2 = H_0^2 a^{-1}, \quad \dot{a} = H_0 a^{-1/2}, \quad \text{ansatz: } a(t) = \left(\frac{t}{t_0}\right)^\alpha$$

$$\frac{\alpha}{t_0^\alpha} t_0^{\alpha-1} = H_0 \left(\frac{t}{t_0}\right)^{-\frac{\alpha}{2}}$$

true if

$$\alpha - 1 = -\frac{\alpha}{2} \Rightarrow \alpha = \frac{2}{3}$$

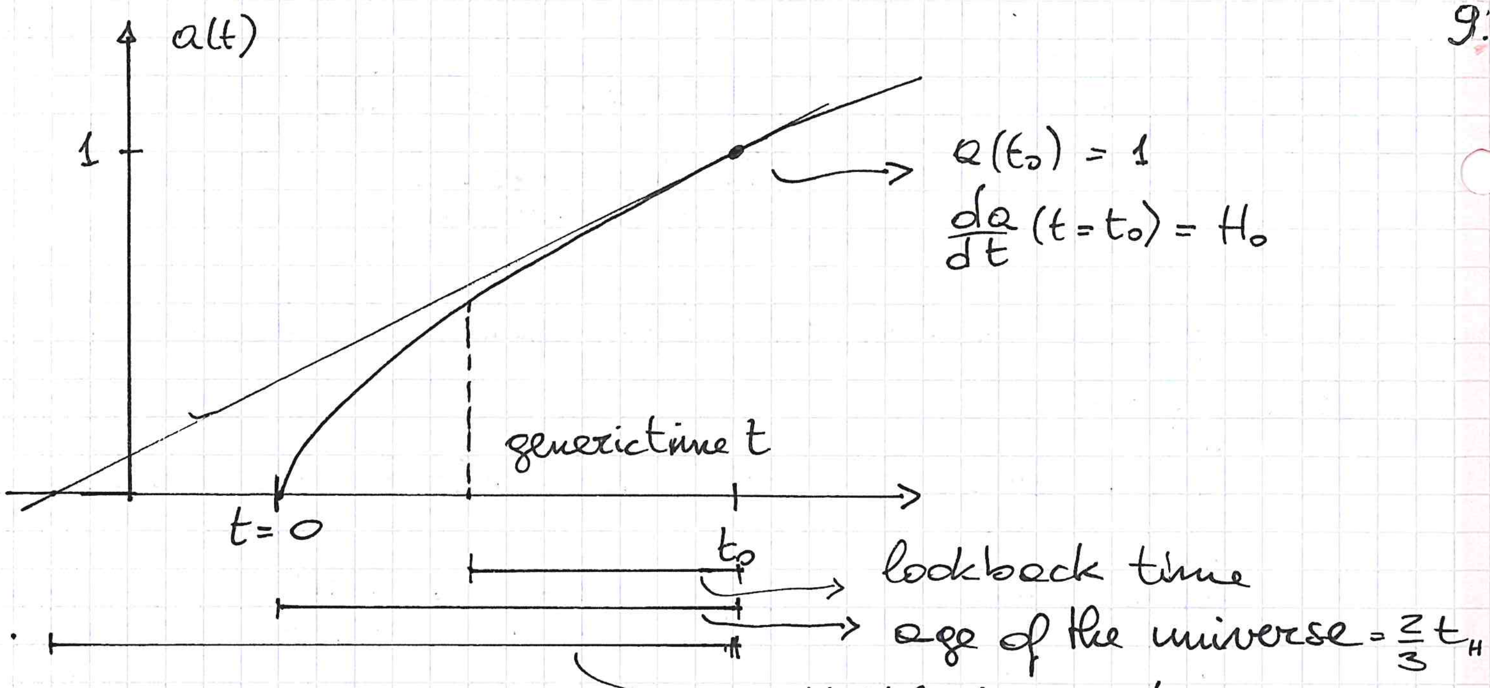
$$\frac{2}{3 t_0^{2/3}} = H_0 t_0^{1/3} \Rightarrow t_0 = \frac{2}{3H_0}$$

$$\boxed{a(t) = \left(\frac{t}{t_0}\right)^{2/3}, \quad t_0 = \frac{2}{3H_0}}$$

t_0 is the time for which $a(t) = 1$, while $a(0) = 0$;
so t_0 is the age of the universe

$t = 0$: $a(0) = 0 \Rightarrow \rho \to \infty$ SINGULARITY

So the Einstein-de Sitter model has a big bang



$$t_H = \frac{1}{H_0} = 9.78\, h^{-1}\ \text{Gyr}$$

Hubble parameter:

$$H(t) = H_0 \left(\frac{\rho}{\rho_{c0}}\right)^{1/2} = H_0\, a^{-3/2} = H_0 \left(\frac{t}{t_0}\right)^{-1} = \frac{2}{3t}$$

Deceleration parameter:

It is given by the ratio of the first two Friedmann eqs:

$$q = -\frac{\ddot{a}\, a}{\dot{a}^2} = -\frac{\ddot{a}}{a}\frac{a^2}{\dot{a}^2} = -\left(-\frac{4\pi G}{3}\rho\right)\frac{3}{8\pi G\rho} = \frac{1}{2}$$

$$\Rightarrow \quad q_0 = q(t=t_0) = \frac{1}{2}$$

Density versus time:

$$\rho = \rho_{c0}\left(\frac{t}{t_0}\right)^{-2} = \frac{3H_0^2}{8\pi G}\frac{1}{t^2}\left(\frac{2}{3H_0}\right)^2 = \frac{1}{6\pi G t^2}$$

or $\quad t = \dfrac{1}{\sqrt{6\pi G\rho}}$, so the cosmic time is the dynamical time of the universe

Distances

COMOVING: $\quad d_c = R_0\chi(z) = \displaystyle\int_t^{t_0}\frac{c\,dt}{a(t)} = c\int_t^{t_0}\left(\frac{t}{t_0}\right)^{-\frac{2}{3}} dt =$

$$= ct_0\int_{t/t_0}^{1} x^{-\frac{2}{3}}\,dx = 3ct_0\left(1-\left(\frac{t}{t_0}\right)^{\frac{1}{3}}\right) =$$

$$= \frac{2c}{H_0}\left(1-\sqrt{a}\right) = \frac{2c}{H_0}\left(1-\frac{1}{\sqrt{1+z}}\right) = \frac{2c}{H_0}\left(\frac{1+z-\sqrt{1+z}}{1+z}\right)$$

PROPER: $d_P = a(t)\, d_c = 3ct_0\left[\left(\frac{t}{t_0}\right)^{2/3} - \frac{t}{t_0}\right]$

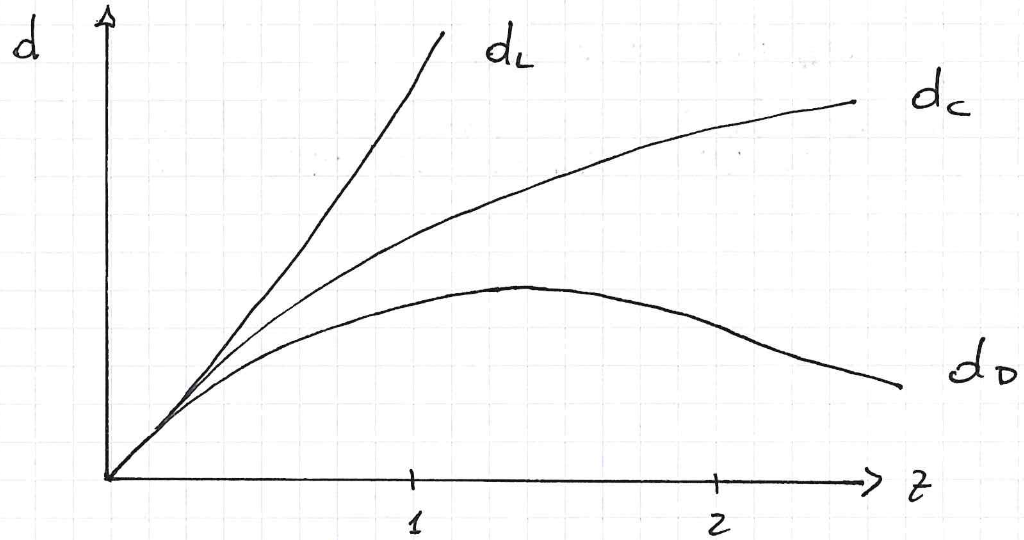
$= \frac{2c}{H_0}\left(a - a^{3/2}\right) = \frac{2c}{H_0}\frac{(1+z-\sqrt{1+z})}{(1+z)^2}$

LUMINOSITY: for a flat space $\chi(r) = r$

$d_L = (1+z) R_0 r = (1+z)\, d_c = \frac{2c}{H_0}(1+z-\sqrt{1+z})$

DIAMETER: again $\chi(r) = r$ in this flat model

$d_D = \frac{1}{(1+z)} R_0 r = \frac{1}{(1+z)}\, d_c = \frac{2c}{H_0}\frac{(1+z-\sqrt{1+z})}{(1+z)^2}$



NB: here $d_D = d_P$

The diameter distance has a maximum:

$\frac{d\, d_D}{dz} = 0 \implies \dots \implies z = \frac{5}{4} = 1.25$

The size of a standard ruler $\vartheta = \frac{D}{d_D}$ would stop decreasing after some redshift z ($\frac{5}{4}$ for Einstein-de Sitter)

This is a classical cosmological test