Cosmology 1

2018/2019
Prof. Pierluigi Monaco

First intermediate test
Topic: general relativity.

This is an extended version of the problem proposed with lecture 10.

After 20 years of travel (according to its own clock), the spaceship described in the Proposed Problem of lecture 2 arrives at the center of the Galaxy, where it meets a well-known black hole, called by astronomers SgrA*, whose mass is $M = 4 \times 10^6$ M$_\odot$ ($M_\odot = 1.99 \times 10^{30}$ kg is the solar mass). With a suitable rocket system, it stops at a coordinate position $r_0 = 1000\, R_s$, in a coordinate system at rest with the black hole for which the metric takes the Schwartzschild form. The spaceship then lets a probe free-fall toward the black hole. The probe sends signals at a given frequency, the spaceship will lose the signal when the frequency will be decreased by a factor of 10. All distances will be expressed either in m or in AU $= 1.50 \times 10^{11}$ m, times will be expressed in s.

(1) How can the spaceship know its "coordinate distance" r from the black hole? try to think of an experiment to determine r.

(2) Fix the values of the orbital parameters $\tilde{E}$ and $\tilde{L}$ in this case, then work out the first-order equations of motion for the probe, starting from $\vec{p}\cdot\vec{p} = -m^2$, where $\vec{p}$ is the probe's momenum. Assuming that the probe has vanishing 3-velocity at $r = r_0$, find $\tilde{E}$ in terms of r_0 and black hole mass M, and write the resulting equation.

(3) Write the geodesic equation for the probe. As a first step, work out the non-vanishing Christoffel symbols from the Schwartschild metric, then, using the probe's proper time τ as an affine parameter, write the 0- and 1-component of the geodesic equation. Demonstrate that the 0-component equation is equivalent to the equation that is obtained by taking the derivative in τ of p^0. Demonstrate that the 1-component is equivalent to the derivative of the equation for $dr/d\tau$ found above.
Hint: remember that $x^\mu \rightarrow (t, r, \vartheta, \varphi)$, $U^\mu = dx^\mu/d\tau$ and that $p_0 = mU_0 = -m\tilde{E}$.

(4) Both the equation for $dr/d\tau$ found on point (2) and the 1-component of the geodesic equation should result remarkably simple and similar to the Newtonian motion of a body under the gravity of a point mass. Following the lecture notes, compute the proper time needed by the probe to get to r=0. (Not strictly required: you can show that it results equal to the so-called dynamical time, $t = \sqrt{3\pi/32G\bar{\rho}}$, where $\bar{\rho} = M/(4\pi r_0^3/3)$.)
Hint: following the notes and expressing $\tilde{E}$ in terms of M and r_0, you should meet an integral of the function $\sqrt{x/(1-x)}$, that is easily solved by assuming $x = \sin^2 t$.

(5) By knowing the 4-velocity of the probe, compute the gravitational redshift of the probe as a function of radius, including the Doppler contribution. You can assume here that the spaceship is at $r \to \infty$, making a small error. Then recompute the radius at which the spaceship loses the signal from the probe, when the observed frequency becomes a factor of 10 lower than that at emission. (Not strictly required: using numerical integration, try to compute the time at which this happens, in the spaceship frame.)

Hint: carefully consider the momentum of the photon $\vec{f}$, knowing that $\vec{f} \cdot \vec{f} = 0$ and that f_0 is conserved. Recall that the photon travels outwards, the probe inwards. Then follow the derivation of gravitational redshift, using the correct $\vec{U}_{\rm pr}$.

Solution

It is useful to recall that the Schwartzschild radius of a $4 \times 10^6\ M_\odot$ black hole is equal to (we explicit the speed of light for this calculation) $R_s = 2GM/c^2 = 1.18 \times 10^{10}$ m $= 0.079$ astronomical units (AU), so $r_0 = 1000 R_s$ correspond to 79 AU.

(1) To determine r it is necessary to perform some physical measurement that depends on r in an invertible way. A simple possibility, that does not require knowledge of black hole mass (that however is known), is to go around the black hole and measure the length of the path. For $\vartheta = \pi/2$ and $dr = d\vartheta = 0$, the space part of the metric is $d\ell^2 = r^2 d\varphi^2$, so $\ell = 2\pi r$. Measuring $2\pi\ell$ is possible if (1) the spaceship knows when it is not moving with respect to the black hole (when it moves, the images of strongly lensed stars move), (2) the spaceship knows when it gets back to the starting point (you can triangulate on distant stars on the other side of the black hole, to avoid lensing). This gives r in physical units, to transform it to R_s requires knowledge of the black hole mass.

Having said this, there are many other physical ways to determine r, any sensible idea will be valid.

(2) In the following I define for simplicity

$$C(r) \equiv 1 - \frac{2GM}{r}$$

Let's call $\vec{p}$ the probe's momentum. The probe falls along a radial geodesic, so $p_\varphi = 0$ and $\tilde{L} = 0$. From $\vec{p} \cdot \vec{p} = -m^2$ we get the equation:

$$\left(\frac{dr}{d\tau} \right)^2 = \tilde{E}^2 - C(r)$$

At $r = r_0$ we have $dr/d\tau = 0$, so

$$\tilde{E}^2 = 1 - \frac{2GM}{r_0} = 1 - \frac{1}{1000} = 0.999$$

and then

$$\left(\frac{dr}{d\tau} \right)^2 = \frac{2GM}{r} - \frac{2GM}{r_0}$$

very similar to the corresponding Newtonian equation (where $\tau \to t$ of course).

(3) The Christoffel symbols have been computed for a spherically symmetric metric, substituting to the ϕ and Λ functions their expressions valid for the Schwartzschil metric one obtains the following non-zero terms (plus symmetric terms in the two lower indices):

$$\Gamma^0_{01} = \frac{GM}{r^2}\frac{1}{C}, \quad \Gamma^1_{00} = \frac{GM}{r^2}C, \quad \Gamma^1_{11} = -\frac{GM}{r^2}\frac{1}{C}$$

$$\Gamma^1_{22} = -rC, \quad \Gamma^1_{33} = -r\sin^2\vartheta\, C, \quad \Gamma^2_{33} = -\sin\vartheta\cos\vartheta$$

$$\Gamma^2_{21} = \frac{1}{r}, \quad \Gamma^3_{31} = \frac{1}{r}, \quad \Gamma^3_{32} = \frac{\cos\vartheta}{\sin\vartheta}$$

Clearly the symbols Γ^2_{33} and Γ^3_{32} vanish for $\vartheta = \pi/2$, while $\Gamma^1_{33} = -rC$.

The 0-component of the geodesic equation results:

$$\frac{d^2t}{d\tau^2} = -2\Gamma^0_{01}\frac{dt}{d\tau}\frac{dr}{d\tau} = -\frac{2GM}{r^2}\frac{1}{C}\frac{dt}{d\tau}\frac{dr}{d\tau}$$

We have that $p^0 = m dt/d\tau = m\tilde{E}/C$; taking the τ-derivative of this equation one can obtain the same equation given above.

The 1-component of the geodesic equation results

$$\frac{d^2r}{d\tau^2} = -\Gamma^1_{00}\left(\frac{dt}{d\tau}\right)^2 - \Gamma^1_{11}\left(\frac{dr}{d\tau}\right)^2 = -\frac{GM}{r^2}C\left(\frac{dt}{d\tau}\right)^2 + \frac{GM}{r^2}\frac{1}{C}\left(\frac{dr}{d\tau}\right)^2$$

Using the expression for $dt/d\tau = \tilde{E}/C$ and $(dr/d\tau)^2 = \tilde{E}^2 - C$ obtained above one can simplify the equation as follows:

$$\frac{d^2r}{d\tau^2} = -\frac{GM}{r^2}$$

again very similar to the Newtonian equation.

(4) The derivation is:

$$\Delta\tau = -\int_{r_0}^{0}\frac{d\tau}{dr}dr = \int_0^{r_0}\frac{dr}{\sqrt{\frac{2GM}{r} - \frac{2GM}{r_0}}}$$

Call $x = r/r_0$:

$$\Delta\tau = \sqrt{\frac{r_0^3}{2GM}}\int_0^1\sqrt{\frac{x}{1-x}}\,dx$$

Now call $x = \sin^2 t$, the integral becomes

$$\int_0^1\sqrt{\frac{x}{1-x}}dx = 2\int_0^{\pi/2}\sin^2 t\,dt = \frac{\pi}{2}$$

Calling now $\bar{\rho} = 3M/4\pi r_0^3 = 1.15\times10^{-6}$ g cm^{-3}, we obtain:

$$\Delta\tau = \sqrt{\frac{3\pi}{32G\bar{\rho}}} = 1.95\times10^6 \text{ s} = 22 \text{ days}$$

It is interesting to note that the proper time needed by the probe to fall from R_s to the singularity can be computed by setting $r_0 = R_s$. It results $\rho = 1157$ g cm^{-3}, $\delta\tau = 7.15\times10^{-4}$ s.

(5) The four-velocity of the probe at radius r is:

$$U^{\mu}_{\text{probe}} = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, 0\right) = \left(\frac{\tilde{E}}{C}, \sqrt{\tilde{E}^2 - C}, 0, 0\right)$$

Calling $\vec{f}$ the photon momentum, and using $\vec{f}\cdot\vec{f} = 0$, it is easy to demonstrate that $f^r = f_0$; of course f_0 is conserved along the photon geodesic. At inifinity, $\vec{U}_\infty = (1, 0, 0, 0)$, so $E_\infty = -\vec{U}_\infty \cdot \vec{f} = -f_0$. Then the photon energy at emission is:

$$E_{\text{em}} = -\vec{U}_{\text{pr}} \cdot \vec{f} = E_\infty \frac{1}{C}\left(\tilde{E} + \sqrt{\tilde{E}^2 - C}\right)$$

Assuming that the spaceship is at inifinity, the gravitational redshift can then be computed as:

$$1 + z = \frac{E_{\text{em}}}{E_\infty} = \frac{1}{C}\left(\tilde{E} + \sqrt{\tilde{E}^2 - C}\right)$$

Contact is lost when $1 + z_{\text{lost}} = 10$, from which it is easy to get (refusing the unphysical solution $C = 0$):

$$C(r_{\text{lost}}) = \frac{2(1 + z_{\text{lost}})\tilde{E} - 1}{(1 + z_{\text{lost}})^2} = 0.1898$$

This is true for

$$r_{\text{lost}} = \frac{R_s}{1 - C(r_{\text{lost}})} \simeq 1.234 R_s$$

The full solution is obtained by considering $E_{\text{obs}} = -\vec{f}\cdot\vec{U}_{\text{ss}} = E_\infty/\sqrt{C(r_0)}$, then the factor to $(1 + z_{\text{lost}})$ in the equation above must be multiplied by $C(r_0)^{-1/2} = 1.0005$. The difference in the result is small.

The coordinate time difference Δt between the probe release and the time it emits the last observable photon at r_{lost} is computed as:

$$\Delta t = -\tilde{E} \int_{r_0}^{r_{\text{lost}}} \left[\left(1 - \frac{2GM}{r}\right)\sqrt{\frac{2GM}{r} - \frac{2GM}{r_0}}\right]^{-1} dr =$$

$$= R_s \int_{1.234}^{1000} \left[\left(1 - \frac{1}{x}\right)\sqrt{\frac{1}{x} - \frac{1}{x_0}}\right]^{-1} dx$$

where $x = r/R_s$. A numerical integration yields a value of ~ 49800 for the integral. However, for $r_{\text{lost}} = 1.234 R_s$ the integrand is only 0.2% larger than the integrand for the proper time $\Delta\tau$, so the time at which the last detectable photon is sent is again $\simeq 22$ days. Indeed, as the divergence of the integral is logarithmic, for this r_{lost} value the integral is still dominated by the time needed by the probe to accelerate from zero to significant speed.

If we want to compute the time at which the spaceship loses contact with the probe, we must add to Δt the time δt required by the photon to get back to the spaceship. This is computed from the metric:

$$\delta t = \int_{r_{\text{lost}}}^{r_0} \frac{dr}{\sqrt{1 - 2GM/r}} = 2GM \int_{r_{\text{lost}}/2GM}^{r_0/2GM} \sqrt{\frac{x}{x-1}}\, dx \simeq 1002 \times 2GM$$

This is just a little larger than the $(1000 - 1.234) \times 2GM$ value one would get for a flat space. The numerical value results $\delta t = 394$ s, negligible with respect to Δt.