Cosmology 1

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First intermediate test
Topic: general relativity.
Deadline: April 23, 13:00.

One year ago the Event Horizon Telescope (EHT) collaboration published the first image of the event horizon of a black hole (BH), M87* at the center of the galaxy M87, located at $D = 16.8$ Mpc (1 pc $= 3.09 \times 10^{16}$ m, 1 Mpc is one million pc), that is known from previous measurements to be as massive as $M = 6.5 \times 10^9$ solar masses $M_\odot$ (1 $M_\odot = 1.99 \times 10^{30}$ kg). The angular resolution of the image is 25 μas (micro-arcseconds). We can think of this image as to a disc of gas that is rotating around the BH almost face-on.

(1) Assuming Euclidean geometry, what is the angular extent of the gravitational radius $R_G = GM/c^2$ as seen from the Earth? would it be possible to image it with the nominal EHT angular resolution? To have an idea of how big it is, compare R_G with the size of the Solar System, then compute the radius of a sphere on the surface of the moon that subtends the same angle as seen from the Earth. These two facts give a feeling of how big this BH is and how far this "nearby" galaxy is.

The angular diameter of the imaged ring is indeed 42 μas. To understand how curvature distorts photon paths we can compute the so-called photon capture radius of a BH, following these steps.

(2) Let's first write down an effective potential for a photon in the Newtonian case. Assuming that b is the photon impact parameter for a point mass M at the center of a cartesian coordinate system in which the photon trajectory lies at $z = 0$ and is aligned with the x-axis, call $x = ct$ its x-coordinate, r its distance from the mass M, and demonstrate that its equation of motion can be written as:

$$\left(\frac{dr}{dt}\right)^2 = c^2 - V_{\rm eff}(r) \qquad (1)$$

Find the expression for the effective potential $V_{\rm eff}$ and interpret it.

(3) Now (using again $c = 1$) write the conserved quantities as $p_0 = -\tilde{E}$ and $p_\varphi = \tilde{L}$, and considering the photon trajectory at very large distances identify $\tilde{E} = h\nu_\infty$ as the photon energy at infinity and $\tilde{L} = b\tilde{E}$ as the photon angular momentum, b being its impact parameter. (*Hint:* let the photon travel at negative y values, so that its motion is counter-clockwise in the $x - y$ plane. Write the cartesian and spherical components of the photon momentum for a flat Minkowski spacetime and for a trajectory at $z = 0$ or $\vartheta = \pi/2$).

(4) Assuming a Schwartzschild metric, and calling λ an affine parameter for the null geodesic of a photon, compute $d^2r/d\lambda^2$ for the photon using the geodesic equation and the condition $d\vec{x}/d\lambda \cdot d\vec{x}/d\lambda = 0$. Find the condition for which the radial acceleration of the photon is positive or negative. How does this result compare with that obtained in class using the effective potential?

(5) From $\vec{p} \cdot \vec{p} = 0$ work out the equation of motion for the photon around the black hole. The photon can escape the black hole if the orbit's pericenter is such to avoid regions with $d^2r/d\lambda^2 < 0$. The pericenter can be computed as the point in which $dr/d\lambda$ changes sign, thus passing through 0. Using the espression for $(dr/d\lambda)^2$ obtain a relation between the orbit pericenter $r_{\rm min}$ and the impact parameter b. Now impose the escape condition worked out in point (4) to obtain a critical impact parameter, called **photon capture radius**. Argue that geodesics can be traveled in both spatial directions in this static metric (outside the event horizon), so that the photon capture radius defines the effective apparent size of the BH event horizon, the BH "shadow".

(6) How does its angular extension for M87* compares with the observed ring?

This is only an order-of-magnitude estimate based on the assumption of a spherically-symmetric, i.e. non-rotating, BH; full ray-tracing of radiation emitted by gas orbiting around a rotating black hole is needed to get a proper prediction.

Solution

1. The gravitational radius of M87* is $R_G = GM/c^2 = 9.59 \times 10^{12}$ m = 63.9 AU (astronomical units). It would contain the whole Solar System, including Kuyper belt. With Euclidian geometry, it would subtents an arc of 3.81 μas at the distance of M87, like a sphere of radius of 0.7 cm on the Moon. This is below the angular resolution limit of EHT.

2. The distance r of the photon, traveling along a straight path $x = ct$, $y = -b$ and $z = 0$, is $r = \sqrt{b^2 + c^2t^2}$. Differentiating it with respect to time, and expressing the result in terms of r, we obtain:

$$\left(\frac{dr}{dt}\right)^2 = c^2 - \frac{c^2b^2}{r^2}$$

So the motion is subject to an effective potential $V_{\text{eff}} = -c^2b^2/r^2$ that has the form of a centrifugal barrier, with cb playing the role of an angular momentum.

3. Using $p^0 = h\nu_\infty$ and $g_{\alpha\beta}p^\alpha p^\beta = 0$, we obtain that the components of the photon momentum in a flat Minkowski spacetime are $(h\nu_\infty, h\nu_\infty, 0, 0)$ for cartesian coordinates and $(h\nu_\infty, h\nu_\infty \cos\varphi, 0, h\nu_\infty \sin\varphi/r)$ in spherical coordinates with $\vartheta = \pi/2$. In this second case the momentum one-form has components $p_0 = -h\nu_\infty$ and $p_\varphi = r\sin\varphi\, h\nu_\infty = bh\nu_\infty$. This motivates the identification of $\tilde{E} = h\nu_\infty$ and $\tilde{L} = bh\nu_\infty = b\tilde{E}$. Note: the positive sign of p_φ is due to the choice of having $y = -b$, so that the photon "orbits" counter-clockwise. Alternatively, $\tilde{L} = -b\tilde{E}$, with no effect on the effective potential where this quantity enters squared.

4. Calling for simplicity $C(r) = 1 - 2GM/r$, the non-vanishing Christoffel symbols for the Schwartzschild metric are:

$$\Gamma^0_{01} = \frac{GM}{r^2}\frac{1}{C(r)}, \quad \Gamma^1_{00} = \frac{GM}{r^2}C(r), \quad \Gamma^1_{11} = -\frac{GM}{r^2}\frac{1}{C(r)}$$

$$\Gamma^1_{22} = -rC(r), \quad \Gamma^1_{33} = -r\sin^2\vartheta\, C(r), \quad \Gamma^2_{33} = -\sin\vartheta\cos\vartheta$$

$$\Gamma^2_{21} = \frac{1}{r}, \quad \Gamma^3_{31} = \frac{1}{r}, \quad \Gamma^3_{32} = \frac{\cos\vartheta}{\sin\vartheta}$$

In the case $\vartheta = \pi/2$, the symbols Γ^2_{33} and Γ^3_{32} vanish while $\Gamma^1_{33} = -rC(r)$. The r-component of the geodesic equation gives:

$$\frac{d^2r}{d\lambda^2} = \frac{GM}{r}\left[-C(r)\left(\frac{dt}{d\lambda}\right)^2 + \frac{1}{C(r)}\left(\frac{dr}{d\lambda}\right)^2\right] + rC(r)\left(\frac{d\varphi}{d\lambda}\right)^2$$

For a null geodesics the tangent vector $d\vec{x}/d\lambda$ is null, so requiring $d\vec{x}/d\lambda \cdot d\vec{x}/d\lambda = 0$ gives:

$$-C(r)\left(\frac{dt}{d\lambda}\right)^2 + \frac{1}{C(r)}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\varphi}{d\lambda}\right)^2 = 0$$

This allows to simplify the geodesic equation as follows:

$$\frac{d^2r}{d\lambda^2} = (r - 3GM)\left(\frac{d\varphi}{d\lambda}\right)^2$$

This shows that for $r < 3GM$ the acceleration is always directed inward, and the photon can only fall into the BH unless it has a positive (and sufficiently large) radial component of the velocity. The radius $r = 3GM$, independent of the photon angular momentum or impact parameter, corresponds to the peak in the effective potential, so this result is entirely consistent with what found during the class.

Note: because the photon momentum is also tangent to the geodetic, one can formulate the geodetic equation in terms of $\vec{p}$ in place of $d\vec{x}/d\lambda$, that corresponds to a specific choice of affine parameter. In this case the equation results:

$$\frac{dp^r}{d\lambda} = \frac{d^2r}{d\lambda^2} = (r - 3GM)\frac{\tilde{L}^2}{r^4}$$

5. The equation of motion of the photon has been discussed in class:

$$\left(\frac{dr}{d\lambda}\right)^2 = \tilde{E}^2 - \left(1 - \frac{2GM}{r}\right)\frac{\tilde{L}^2}{r^2}$$

Using $\tilde{L} = b\tilde{E}$, the radial component has a minimum when $dr/d\lambda = 0$, that happens at a radius $r_{\rm min}$ such that $b^2(r_{\rm min} - 2GM) = r^3_{\rm min}$. Setting $r_{\rm min} = 3GM$ allows to obtain the smallest impact parameter for which the photon can escape back to infinity:

$$b_{\rm min} = \sqrt{27}\,GM = 5.20\,R_G$$

Indeed, a photon traveling from infinity with a smaller impact parameter would get inside $3GM$ with a negative radial velocity, and would then fall toward the singularity.

The subtle part here is to argue that the spatial path of the geodesic can be traveled in both directions. The following discussion is based on the discussion we have had in class. A geodesic is a curve, and as such can in principle be traveled in both directions; but particles can only travel it forward in time. Reverting the sign of the affine parameter λ means to revert the direction of the tangent vector, but all affine parameters give the same geodesic and then the same dynamics. So reverting the sign of λ does not help. Let's say a photon travels along a geodesic that connects events $A = \{t_A, x^i_A\}$ and $B = \{t_B, x^i_B\}$, with $t_B > t_A$ (here the pedices are not coordinate indices). A photon that travels in reverse direction would connect the events $A' = \{t_{A'}, x^i_A\}$ and $B' = \{t_{B'}, x^i_B\}$, with $t_{A'} > t_{B'}$; A and A' have the same space coordinates, as well as B and B' do. The question is if the geodesics that connects $A - B$ and $A' - B'$ have the same space path. We can obtain the two geodesics inverting the sign of time coordinate t. All the Christoffel symbols do not depend on time, and the only non-vanishing symbol that includes $dt/d\lambda$ and appears on one of the space geodesic equations is Γ^1_{00}, so $dt/d\lambda$ enters squared in this equation. Then the equations of space geodesics will be the same and describe the same path. This is not true for the time geodesic equation, where the Γ^0_{01} symbol will lead to a term like $dr/d\lambda\, dt/d\lambda$, so the source of the equation changes sign.

So photons collected by a very distant telescopes will avoid those geodesics that would lead a photon traveling from us toward M87 to fall inside the event horizon, and the event horizon itself would be seen as a shadow of apparent size $\sqrt{27}GM$.

6. The photon capture radius of M87* is $\sqrt{27} = 5.20$ times the gravitational radius, so the expected apparent diameter of the ring is $2\sqrt{27}GM/D$ ($c = 1$), amounting to 39.6 μas, not very different from the measured value of 42 μas.