Cosmology 1

2020/2021
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First intermediate test
Topic: general relativity.
Deadline: April 29, 11:00.

A human-made probe of mass m is in a circular orbit at radius R around a Schwartzschild black hole of mass M; as an example, this could be SgrA*, whose mass is $M = 4 \times 10^6$ M$_\odot$ ($M_\odot = 1.99 \times 10^{30}$ kg is the solar mass). The probe sends a radio signal at the end of each orbit. This periodic signal is received by a distant observer at rest in the reference frame where the metric has the Schwartschild form.

(1) How can the probe know that one orbit has been completed and it's time to send the signal? elaborate on this question.

(2) Solve for the $\tilde{E}$ and $\tilde{L}$ parameters in terms of R, and express the four-velocity of the probe.

(3) Compute the proper-time duration of the orbit (in s) as a function of R. How does this compare to Kepler's third law? Then assume that $R = 6GM$ or $R = 12GM$, and quantify the result for SgrA*.

(4) Perform a similar calculation in terms of coordinate time. Argue that this is the time measured by the observer at infinity. Compare again the analytic result to Kepler's third law, and quantify it in the case of SgrA* for $R = 6GM$ and $R = 12GM$.

(5) An Active Galactic Nucleus (AGN) is associated with a supermassive black hole that is accreting mass through an accretion disc. Matter is then orbiting in nearly circular orbits down to the last stable orbit, slowly losing angular momentum; infall causes matter to heat up and to emit electromagnetic radiation. Starting from the circular orbits considered above, suppose that an AGN emission were varying periodically (it's a highly idealistic case!) with a period of 25 minutes, how could this fact be translated into a constraint on the black hole mass?

As a general suggestion, it may be useful to express the orbit radius in units of the gravitational radius, $R = aGM$.

Solution

(1) Assume that the probe has a camera that can be pointed in any direction and a computer that can elaborate images. The probe can infer the radial direction because it can image the angular extent of the event horizon (the black hole "shadow"), and reconstruct the position of the singularity. The safest thing to do is to image the stars opposite to the singularity, where light arrives through radial null geodesics. Then it will be sufficient to match the image to a star map to know the probe angular position. A convenient choice may be to let the brightest star that passes through the imaged region trigger the signal.

(2) A circular orbit has minimum energy for a given angular momentum, so fixing $\tilde{L}$ one can obtain the circular orbit radius R_c by setting $d\tilde{V}^2/dr = 0$. This leads to:

$$R = \frac{\tilde{L}^2}{2GM}\left(1+\sqrt{1-\frac{12(GM)^2}{\tilde{L}^2}}\right)$$

provided that $\tilde{L}^2 > 12(GM)^2$. Here we take the solution at largest radius, that represents the minimum of the potential. This relation can be inverted to obtain $\tilde{L}$:

$$\tilde{L}^2 = \frac{GMR^2}{R-3GM} = (GM)^2\frac{a^2}{a-3}$$

with $R \geq 6GM$; the second relation has been obtained assuming $R = aGM$, $a \geq 6$. For a circular orbit, $dr/d\tau = 0$, so $\tilde{E}^2 = \tilde{V}^2$:

$$\tilde{E}^2 = \frac{(R-2GM)^2}{R(R-3GM)} = \frac{(a-2)^2}{a(a-3)}$$

The four-momentum of the probe can be easily recovered by raising the index of p_μ, knowing that $p_0 = -m\tilde{E}$ and $p_\varphi = m\tilde{L}$, while $p_r = 0$ for a radial orbit and $p_\vartheta = 0$ when the orbit is at $\theta = \pi/2$. The four-velocity is $U^\mu = p^\mu/m$, so:

$$\vec{U} = \left(\sqrt{\frac{R}{R-3GM}}, 0, 0, \frac{1}{R}\sqrt{\frac{GM}{R-3GM}}\right) =$$

$$= \left(\sqrt{\frac{a}{a-3}}, 0, 0, \frac{1}{aGM}\sqrt{\frac{1}{a-3}}\right)$$

(3) The proper time is obtained by integrating $d\tau = d\tau/d\varphi \cdot d\varphi = 1/U^\varphi \cdot d\varphi$ in φ from 0 to 2π. Because U^φ does not depend on φ, the integration gives a 2π factor:

$$\Delta\tau = \frac{2\pi}{U^\varphi} = 2\pi GM\sqrt{\frac{R^3}{GM^3}\left(1-\frac{3GM}{R}\right)} = 2\pi GM\sqrt{a^3\left(1-\frac{3}{a}\right)}$$

Third Kepler's law predicts that $\Delta\tau^2 \propto R^3$, so we find here a correction of $(1-3/a)$ to $\Delta\tau^2$, that vanishes for $a \to \infty$. For $a = 6$ (the last stable orbit) and $a = 12$ and for SgrA*, with explicit c, we obtain:

$$\Delta\tau(a=6) = 12\pi\sqrt{3}\,\frac{GM}{c^3} = 1284\,\mathrm{s} = 21.4\,\mathrm{min}$$

$$\Delta\tau(a=12) = 72\pi\,\frac{GM}{c^3} = 4448\,\mathrm{s} = 74.1\,\mathrm{min}$$

(4) An observer at infinity, stationary in our coordinate system, has four-velocity $\vec{U} = (1,0,0,0)$ and its metric reduces to Minkowski, so time in its frame is the coordinate time. To compute it we must integrate $dt = dt/d\tau \cdot d\tau/d\varphi \cdot d\varphi = U^0/U^\varphi \cdot d\varphi$ in φ from 0 to 2π:

$$\Delta t = 2\pi\frac{U^0}{U^\varphi} = 2\pi\sqrt{\frac{R^3}{GM}} = 2\pi GMa^{3/2}$$

So in this case we recover third Kepler's law. For SgrA* we have:

$$\Delta\tau(a=6) = 12\pi\sqrt{6}\,\frac{GM}{c^3} = 1815\,\mathrm{s} = 30.3\,\mathrm{min}$$

$$\Delta\tau(a=12) = 48\pi\sqrt{3}\,\frac{GM}{c^3} = 5136\,\mathrm{s} = 85.6\,\mathrm{min}$$

It is possible to see that $\Delta\tau$ and Δt are different by 40% at $a = 6$, but only by 15% at $a = 12$.

(5) The variability of AGN luminosity can be used to constrain the black hole mass. In our very simplified case of a circular orbit, we assume that the observed variability (periodicity in our case) is due to some light-emitting object orbiting around the black hole at $R \geq 6GM$, or $a \geq 6$. Then from $\Delta t = 2\pi a^{3/2}GM = 25$min, the condition $a \geq 6$ leads to a lower limit on the mass (with explicit c):

$$M \leq \frac{c^3\Delta t}{2\pi 6^{3/2}G} = 3.3 \times 10^6\, M_\odot$$

A true mass estimate requires some knowledge of a. This idea cannot be used in practice, but an even more interesting argument can be made based on AGN variability: suppose we do not assume that the central object is a black hole, and we have prior knowledge on its mass; then the timing can be used to estimate a or R. Then the central object, if it is not a black hole, must have a radius $< R$, to allow for something to orbit around it at that radius. The closer is a to values or order unity, the more compact the object must be, and the more difficult is to hold such a compact object against its own gravity, especially for such large black hole masses. So variability is considered as an argument in favour of the hypothesis that AGNs contain a supermassive black hole.