Cosmology 1

2022/2023
Prof. Pierluigi Monaco

First intermediate test
Topic: general relativity.
Deadline: April 13, 11:00.

This exercise is based on Problem N. 10 and the first test of 2018/2019.

To directly measure the mass of a nearby stellar black hole, that lies at 10 pc from us, future astronomers decide to launch a probe that falls into the black hole. We will assume that the black hole does not rotate and the probe travels in a perfectly radial orbit that has null velocity at infinity. The probe sends a signal at a constant frequency of 30 GHz. A receiver on Earth, that for simplicity will be assumed to be an observer at infinity that is at rest with the black hole, receives the signal and records its frequency with a cadence of 1 ms. The last 20 ms of data (frequencies in GHz) before the signal is lost are the following:

1	25.56 ± 0.71	6	23.47 ± 0.97	11	22.03 ± 0.84	16	20.61 ± 0.85
2	24.15 ± 0.92	7	23.73 ± 1.00	12	22.28 ± 0.92	17	19.47 ± 1.06
3	26.20 ± 1.02	8	22.88 ± 0.83	13	24.23 ± 1.00	18	17.13 ± 0.79
4	23.65 ± 0.64	9	25.62 ± 0.96	14	21.50 ± 1.10	19	14.33 ± 1.14
5	25.02 ± 0.83	10	23.78 ± 1.10	15	20.62 ± 1.08	20	4.53 ± 0.90

The aim is to infer the mass $M_\bullet$ of this black hole, including an errorbar. To achieve this goal you can follow this procedure.

(1) From $\vec{p} \cdot \vec{p} = -m^2$, where m is the mass of the probe, compute $dr/d\tau$ for the probe.

(2) Following the first test of 2018/2019, compute the observed frequency received on Earth, including gravitational and Doppler redshifts.

(3) Use the probe four-velocity to compute $dt/d\tau$.

(4) Don't forget that the photon emitted by the probe must travel back to the Earth, you will need to know the distance traveled by the probe $\ell(\tau)$

You will obtain a set of equations that can be numerically integrated in τ, the affine parameter of the probe's geodesic. It is very convenient to use adimensional quantities, dividing lengths by $R_g = GM_\bullet/c^2$, the gravitational radius of the black hole, and times by R_g/c, the light crossing-time of the gravitational radius. Integrate the solution starting from a relatively large distance, say $200R_g$, until you reach $r = 2R_g$. Adding dimensions to your adimensional quantities, you will be able to scale your model with $M_\bullet$.

When comparing the model with the data, you have freedom to set the zero of the time scale. One possible choice is to set $t = 0$ when the gravitational redshift reaches $1 + z = 10$, so that the observed frequency is 1/10th of the initial frequency; you can use the last observation, that is consistent with it, as your zero point. This way, a simple χ^2 fit can give you the solution. It is clear that other equivalent solutions may work as well.

Alternatively (this would be appreciated though it is not strictly required), you can add a constant to the time scale (a "nuisance parameter"), and fit the data with a model with two parameters. The measurement will be given by the marginalization of the 2D probability over the nuisance parameter, i.e. by projecting the confidence ellipsis on the $M_\bullet$ axis.

Give a quick discussion of the results, but **remember that the solution must not be longer than 4 pages plus figures**. In this discussion quickly reply to these answers:

(1) what is the coordinate distance r of the probe when it sends it last signal?

(2) what is its velocity at the same time?

(3) seen from the Earth, the probe should take an infinite time to fall into the black hole, what does it take to see this effect?

(4) how can we test from Earth the hypothesis that the orbit is radial?

Solution

Let's first find a model for the black hole mass.

(1) The computation of $dr/d\tau$ can be found in the lecture notes and in the textbook. We need to find the $\tilde{E}$ and $\tilde{L}$ parameters, that define the orbit energy and angular momentum. A radial orbit has vanishing p_φ, so $\tilde{L} = 0$. The equation of motion of the orbit can be written simply as:

$$\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - \left(1 - \frac{2GM_\bullet}{r}\right)$$

We require the velocity $dr/d\tau$ to vanish at inifinity, and this is true if $\tilde{E} = 1$, leaving:

$$\frac{dr}{d\tau} = -\sqrt{\frac{2GM_\bullet}{r}}$$

Here we choose the negative solution of the square root because the probe is infalling into the black hole.

(2) Let's call f_α the four-momentum one-form of a photon that travels toward the Earth in a radial orbit. Its f_ϑ and f_φ components will vanish, and its radial and time components will be connected by $f_\alpha f_\beta g^{\alpha\beta} = 0$, leading to:

$$f_r = \frac{1}{1 - 2GM_\bullet/r} f_0$$

The energy of the photon in the frame of the probe, $h\nu_{\rm em}$, is:

$$h\nu_{\rm em} = -U^\alpha f_\alpha$$

where U^α is the probe four-velocity. This can be worked out following the lecture notes:

$$U^\alpha = \left(\frac{1}{1 - 2GM_\bullet/r}, \sqrt{2GM_\bullet/r}, 0, 0\right)$$

This allows us to relate $h\nu_{\rm em}$ to f_0. For a distant observer with four-velocity $(1, 0, 0, 0)$, $f_0 = -h\nu_{\rm obs}$, leading to:

$$1 + z = \frac{h\nu_{\rm em}}{h\nu_{\rm obs}} = \frac{1 + \sqrt{2GM_\bullet/r}}{1 - 2GM_\bullet/r}$$

(3) To relate the proper time of the probe to the coordinate time t, that is the time for the distant observer, we need:

$$\frac{dt}{d\tau} = U^0 = \frac{1}{1 - 2GM_\bullet/r}$$

(4) The photon emitted by the probe must also travel back to the distant observer; the timing of the measurements will be the coordinate time of emission plus the travel time of the photon. It is not necessary to integrate this for 10 pc, one can compute the distance to a point that is several gravitational radii away from the black hole, so that its clock ticks (almost) at the same rate as the distant observer. For a photon the travel

time is equal to the travelled distance, divided by the speed of light if this is not unity. The distance can be computed as follows:

$$d\ell = \sqrt{g_{rr}}\,\frac{dr}{d\tau}\,d\tau = \sqrt{\frac{2GM_\bullet/r}{1-2GM_\bullet/r}}\,d\tau$$

where we choose to integrate in proper time.

Calling $x = \tau/GM_\bullet$, $y = r/GM_\bullet$, $T = t/GM_\bullet$ and $L = \ell/GM_\bullet$, we can create a model for our data by integrating this set of differential equations:

$$\frac{dy}{dx} = -\sqrt{\frac{2}{y}} \quad (1)$$

$$\frac{dT}{dx} = \frac{y}{y-2} \quad (2)$$

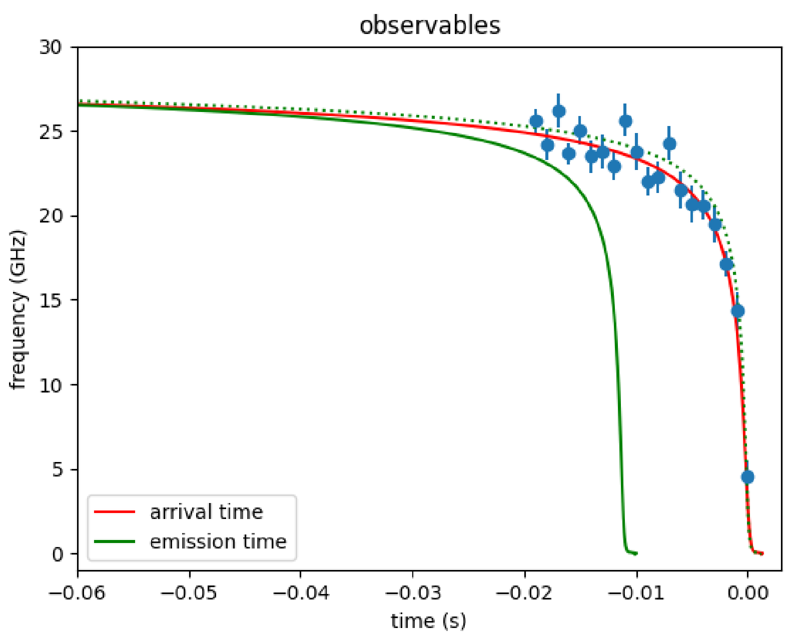
$$\frac{dL}{dx} = \sqrt{\frac{2}{y-2}} \quad (3)$$

To these equations we add:

$$\nu_{\rm obs} = \nu_{em}\frac{y-2}{y+\sqrt{2y}} \quad (4)$$

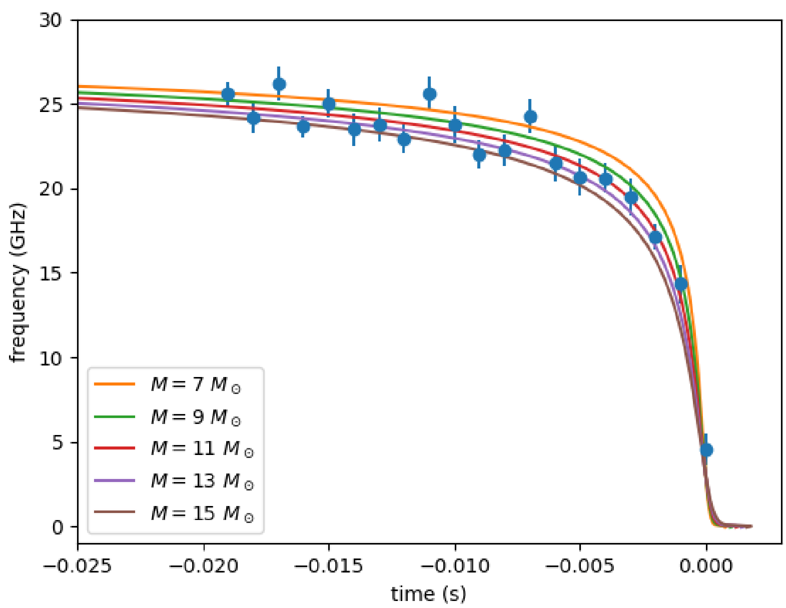
These equations can be integrated numerically, starting from $y \gg 1$ (we use $y = 200$) and progressing in proper time until it happens that $y \leq 2$.

This model can be computed once and then adapted to any $M_\bullet$. When translating x, y, T and L to physical quantities, we plug back the speed of light c, so that time variables are multiplied by $R_g/c = GM_\bullet/c^3$ and length variables by $R_g = GM_\bullet/c^2$.



We start by setting $t = 0$ as the time at which $1+z = 10$, so the frequency is 3 GHz; the last measured point is consistent with 3 within 2σ (1.7σ difference), so we take it as a proxy for setting $t = 0$. Of course this is only a specific choice, setting $t = 0$ when $1 + z = 30/4.53 \simeq 6.62$ would have been an equally good choice. The figure above shows how a model with 11.2 $M_\odot$ fits the data (this is the true value used to generate the data). To show the effect of adding the photon travel time, the green lines shows what happens if one uses the coordinate time at photon emission instead of the arrival time; the continuous line has the same time definition as the red line, for the dotted line we defined $t = 0$ as the (emission) time at which $1 + z = 10$.

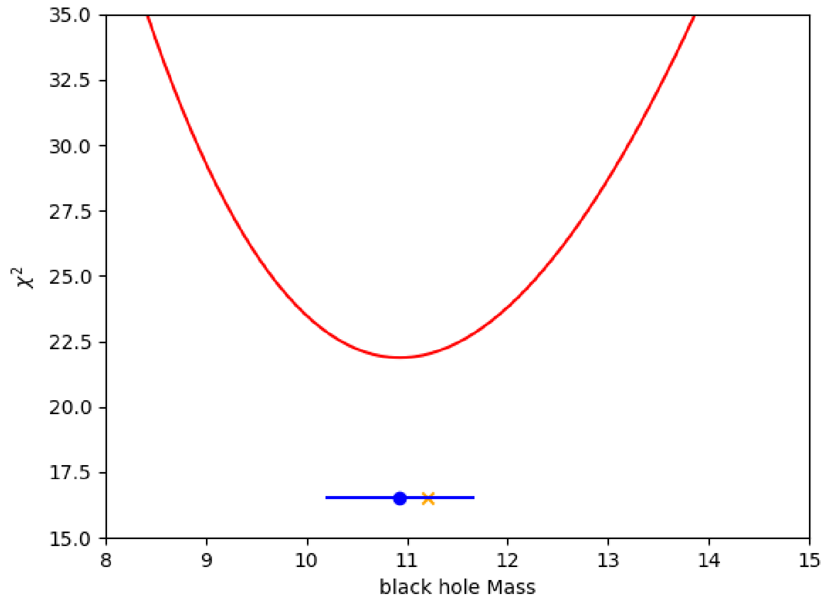
In these figures the flattening of the curve at late times and very low frequencies corresponds to the probe approaching the event horizon at infinite coordinate time; one can stretch this curve toward larger and larger times by integrating the equations at higher and higher precision. Clearly, this phase is not probed by our data.



The figure above shows how the prediction changes with the black hole mass $M_\bullet$. This gives a first idea on the errorbar one can obtain. This 1-parameter model can be fit on the data by simply minimizing the χ^2. The figure below shows the reduced χ^2 as a function of $M_\bullet$, the point with errorbar below shows the resulting value:

$$M_\bullet = 10.9 \pm 0.8$$

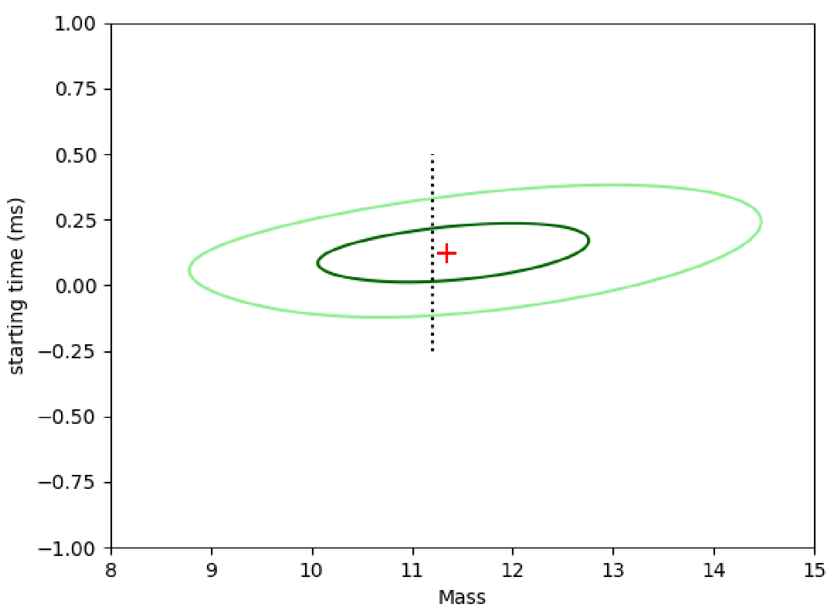
fully consistent with the true value of $M_\bullet = 11.2\ M_\odot$ (denoted by a cross in the figure).



We then add a "nuisance parameter" by allowing a time shift δt between the model predictions and the observational data. This way the model has two parameters. The figure below shows the resulting confidence intervals, defined as the levels of reduced χ^2 at $\chi^2_{\mathrm{min}} + 1$ and $\chi^2_{\mathrm{min}} + 2$. Projecting the inner confidence interval on the two axes, one obtains measures for $M_\bullet$ and δt:

$$M_\bullet = 11.35 \pm 0.83$$

$$\delta t = 0.00012 \pm 0.00006$$



The parameter δt is consistent with zero within 2 σ; the measurement of $M_\bullet$ is again fully consistent with the true one (denoted by the dotted line in the figure), though a slight degeneration between the two parameters makes its value a bit larger. The measurement has also a slightly larger errorbar; this is a typical consequence of fitting data with a model with more parameters. Conversely, the minimum reduced χ^2 passees from 1.15 to 1.04, showing the convenience of adding this nuisance parameter.

The final four questions can be aswered as follows.

(1) By reverting eq. 4, and propagating the measurement errors of the frequency (but using the true mass, one should propagate also the error in the measurement of $M_\bullet$), we can easily see that when the probe is last detected it has $1+z = 6.61 \pm 1.37$ and a coordinate radius of $y = 2.77 \pm 0.20$ gravitational radii. This is well within both the last stable orbit and the photon radius.

(2) As a speed, we report $dr/d\tau = -c\sqrt{2/y}$ of the probe at the observation time; it results $v = 254600 \pm 9000$ km/s, or $v/c = 0.849 \pm 0.030$.

(3) The probe takes an infinite time to approach the event horizon, but as shown in the figures above this slow-down would only be observed if we were able to detect radiation at frequencies below 1 GHz ($z > 30$).

(4) If the orbit is radial, we should see the probe's light always coming from the same point on the sky. Assuming that $M_\bullet = 11.2\ M_\odot$, the gravitational radius is $R_g = 16.5$ km, subtending an angle of only 5.34×10^{-14} radians, that is $\sim 1.1 \times 10^{-8}$ arcsec. Gravitational lensing helps in increasing the black hole shadow of the Schwartzschild radius $2R_g$ by a factor $\sim \sqrt{27}$, leading to a factor 10.4 increase of the angle. Clearly this angular resolution requres extreme interferometry to be achieved: the wavelength corresponding to 30 GHz is $\lambda = 1$ cm, so we need a baseline such that $\lambda/D \sim 5 \times 10^{-13}$, leading to $D \sim 2 \times 10^{12}$ cm, that is 0.13 AU, 13% of the Sun-Earth distance. One could argue that a technology able to send a probe to a nearby black hole can build such a large orbiting radio telescope.