Cosmology 1

2023/2024
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First intermediate test
Topic: general relativity.
Deadline: March 25, 11:00.

A spaceship travels to SgrA* ($M = 4.1 \times 10^6$ $\mathrm{M}_\odot$) to experiment the twin paradox. The ship sets on a circular orbit at $r \gg 2GM/c^2$, so that we can approximate its proper time as the coordinate time t. Two twins synchronize their clocks, then one travels on a probe to a circular orbit at $R > 2GM/c^2$ but not much larger. The probe orbits N times around the black hole, then flies back to the spaceship. We will assume that SgrA* is not rotating and will neglect the time needed by the probe to go to its orbit and then fly back; this latter condition is obtained is N is large, so we can start by assuming $N \sim 100$.

(1) Suggest a procedure to let the probe measure R and N during its orbit.

(2) Compute, as a function of R and N, how much younger is the twin traveling near the black hole when they reunite. Quantify these time differences for relevant values of R and N.

(3) Discuss how to choose R and N for this experiment.

Solution

The calculation of proper-time and coordinate-time period of a radial orbit of a massive object around a black hole has already been discussed in the first intermediate test of 2021. Here we report the main calculations, where we assume $c = 1$.

A circular orbit has minimum energy for a given angular momentum, so fixing $\tilde{L}$ one can obtain the circular orbit radius R by setting $d\tilde{V}^2/dr = 0$. This leads to:

$$R = \frac{\tilde{L}^2}{2GM}\left(1 + \sqrt{1 - \frac{12(GM)^2}{\tilde{L}^2}}\right)$$

provided that $\tilde{L}^2 > 12(GM)^2$. Here we take the solution at largest radius, that represents the minimum of the potential. This relation can be inverted to obtain $\tilde{L}$:

$$\tilde{L}^2 = \frac{GMR^2}{R - 3GM} = (GM)^2 \frac{a^2}{a-3}$$

with $R \geq 6GM$; the second relation has been obtained assuming $R = aGM$, $a \geq 6$. For a circular orbit, $dr/d\tau = 0$, so $\tilde{E}^2 = \tilde{V}^2$:

$$\tilde{E}^2 = \frac{(R - 2GM)^2}{R(R - 3GM)} = \frac{(a-2)^2}{a(a-3)}$$

The four-momentum of the probe can be easily recovered by raising the index of p_μ, knowing that $p_0 = -m\tilde{E}$ and $p_\varphi = m\tilde{L}$, while $p_r = 0$ for a radial orbit and $p_\vartheta = 0$ when the orbit is at $\theta = \pi/2$. The four-velocity is $U^\mu = p^\mu/m$, so:

$$\vec{U} = \left(\sqrt{\frac{R}{R - 3GM}}, 0, 0, \frac{1}{R}\sqrt{\frac{GM}{R - 3GM}}\right) =$$

$$= \left(\sqrt{\frac{a}{a-3}}, 0, 0, \frac{1}{aGM}\sqrt{\frac{1}{a-3}}\right) = \left(\frac{dt}{d\tau}, 0, 0, \frac{d\varphi}{d\tau}\right)$$

The proper time of a complete orbit is obtained by integrating $d\tau = d\tau/d\varphi \cdot d\varphi = 1/U^\varphi \cdot d\varphi$ in φ from 0 to 2π. Because U^φ does not depend on φ, the integration gives a 2π factor:

$$\tau_{\text{orb}} = \frac{2\pi}{U^\varphi} = 2\pi GM\sqrt{\frac{R^3}{GM^3}\left(1 - \frac{3GM}{R}\right)} = 2\pi GM\sqrt{a^3\left(1 - \frac{3}{a}\right)}$$

The coordinate time is computed by integrating $dt = dt/d\tau \cdot d\tau/d\varphi \cdot d\varphi = U^0/U^\varphi \cdot d\varphi$ again in φ from 0 to 2π:

$$t_{\text{orb}} = 2\pi\frac{U^0}{U^\varphi} = 2\pi\sqrt{\frac{R^3}{GM}} = 2\pi GMa^{3/2}$$

To express these quantities with explicit c, it is sufficient to notice that a time is obtained by substituting GM with GM/c^3, the light-crossing time of the gravitational radius. For SgrA*, $GM/c^3 = 20.2$ s.

These are the answers to the questions raised above:

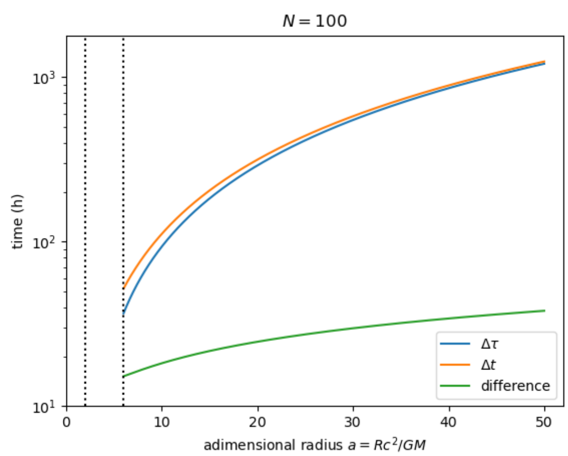
(1) Of course there are several procedures one can adopt to measure R and N. Once the circular orbit at $R = aGM$ is reached, one can use the distant stars to understand how many times the probe has revolted around the black hole, but it will be useful to use stars that are opposite to the singularity to avoid lensing. For a Schwartzschild black hole the position of the singularity can be worked out by finding the center of the opening of the event horizon, that will be completely dark. To measure R, one can assume that a technological society able to send a spaceship around a black hole knows the black hole mass with great accuracy, so one can reverse the relation between the orbital time $\tau_{\rm orb}$ and R. Alternatively, precise timing of light signals from the spaceship to the probe can be used to recover the same quantity, although a precise ray tracing model will be needed to exploit this information.

(2) The proper-time and coordinate-time durations of the experiment,

$$\Delta\tau = N\tau_{\rm orb} = 2\pi N(a^3(1-3/a))^{1/2}\, GM/c^3$$

$$\Delta t = N t_{\rm orb} = 2\pi N a^{3/2}\, GM/c^3$$

are shown in the figure below as a function of a, for $N = 100$. Dotted vertical lines mark the event horizon and the Innermost Stable Circular Orbit (ISCO), $a = 6$. Times scale proportional to N, so it is easy to rescale these numbers to a generic value.



The ratio $\Delta\tau/\Delta t = (1-3/a)^{1/2}$ decreases with decreasing a, so the relative effect of curvature is greater when the twin on the probe travels nearer to the event horizon, and has a maximum value of $\sqrt{1/2} \simeq 0.707$ at the ISCO. However, because the more distant orbits have longer periods, the total time over at fixed N grows with a. These are the values of $\Delta\tau$, Δt and their difference, in hours, for some values of R and N.

	$a = 6$	$a = 10$	$a = 20$	$a = 50$
$N = 50$				
$\Delta\tau$	18.3	46.6	145.3	604.1
Δt	25.9	55.7	157.6	623.1
difference	7.6	9.1	12.3	19.0
$N = 100$				
$\Delta\tau$	36.6	93.3	290.7	1208.2
Δt	51.8	111.5	315.3	1246.2
difference	15.2	18.2	24.6	38.0
$N = 200$				
$\Delta\tau$	73.3	186.5	581.3	2416.4
Δt	103.6	222.9	630.5	2492.3
difference	30.3	36.4	49.2	75.9

(3) While the relative effect is stronger when the probe gets near the ISCO, the cumulative effect at fixed N is larger when the probe orbits at larger distances, so getting very near the even horizon is not mandatory. Moreover, all time differences amount to at least several hours, a time interval that can be measured very accurately. One can imagine that the details of the experiment will be determined by a compromize of minimizing the risk (staying away from the ISCO) and not asking to a human to stay in a small probe for a very long time. But probably the best option will be to send an automated probe with a clock, so the two twins can have a spritz while they wait for the probe to come back.