Cosmology 1

2024/2025
Prof. Pierluigi Monaco

First intermediate test
Topic: general relativity.
Deadline: March 25, 11:00.

A future, hyper-technological humanity decides to collapse the Moon into a black hole, so as to solve at the same time the energy problem and the toxic waste storage problem, with minimal effects on sea tides. As GR experts you are asked to characterize the resulting black hole and the trajectory of a packet of toxic waste that is to be accreted on the black hole Moon. You can assume that a packet of mass $m = 1000$ kg, after leaving the Earth, travels towards the black hole Moon at a speed v equal to the escape velocity from the planet with some impact parameter b, and that outside the Earth gravity is dominated by the black hole.

(1) Compute the gravitational, Schwartzschild, photon and ISCO radii and their angular extent as seen from the Earth if Euclidean geometry is valid, in arc-seconds; for the photon radius give also its apparent angular size including curvature.

(2) To compute the impact parameter needed to launch a packet into the black hole Moon, first write the momentum of this packet at infinite distance from the black hole moon, and identify the constants $\tilde{E}$ and $\tilde{L}$ in terms of mass m, impact parameter b, velocity v and Lorentz factor γ.

(3) Compute the largest impact parameter that would allow the toxic packet to fall directly into the black hole Moon, reporting its value in physical units and in arcsec.

(4) Compute the smallest impact parameter that would allow the packet to reach an accretion disc at its pericenter.

(5) What strategy would you suggest to launch packets to the black hole Moon and extract energy? keep it short! the report should not be longer than four pages.

Solution

We use for the Moon mass $M = 7.346 \times 10^{25}$ g, for the distance from the Earth surface 3.78×10^{10} cm; for the Earth mass and radius, respectively, 5.9724×10^{27} g and 6.378×10^{8} cm.

(1) We report in this table the various required radii; the table includes quantities that will be defined below.

	physical	angular
$R_g = GM/c^2$	54.6 μm	2.98×10^{-8} arcsec
$R_s = 2R_g$	109.1 μm	5.95×10^{-8} arcsec
$R_{\rm ph,E} = 3R_g$	163.7 μm	8.93×10^{-8} arcsec
$R_{\rm ph,GR} = \sqrt{27}R_g$	283.5 μm	1.55×10^{-7} arcsec
$R_{\rm isco} = 6R_g$	327.3 μm	1.79×10^{-7} arcsec
$b_{\rm max}$	5.07 m	2.77×10^{-3} arcsec
$b_{\rm min}$	6.21 m	3.39×10^{-3} arcsec

Clearly, the size of the black hole Moon is exceedingly small, and almost impossible to be resolved from the Earth.

(2) It is very easy to be "far" from the black hole Moon, if far means to be at many R_g; instead, assuming that the metric can be approximate as Minkowski between the Earth and the Moon is a clearly oversimplification. Nonetheless, in this approximation we can write the momentum of the packet in Cartesian coordinates as:

$$p^\mu = (E, p, 0, 0) = (m\gamma, m\gamma v, 0, 0)$$

and in spherical coordinates, calling $b := r \sin\varphi$:

$$p^\mu = (E, p\cos\varphi, 0, p\sin\varphi) = (m\gamma, m\gamma v\cos\varphi, 0, m\gamma v b/r^2)$$

and

$$p_\mu = (-m\gamma, m\gamma v\cos\varphi, 0, m\gamma v b)$$

Then if $\tilde{E} = -p_0/m$ and $\tilde{L} = p_\varphi/m$ we have:

$$\tilde{E} = \gamma, \qquad \tilde{L} = \gamma v b$$

In other words, $\tilde{E}$ is the Lorentz gamma factor of the toxic waste packet after exiting Earth and $\tilde{L}$ its specific angular momentum with respect to the Moon. If the packet travels at the the Earth escape velocity (that should be correct at the order-of-magnitude level), then:

$$v = \sqrt{\frac{GM_{\rm earth}}{R_{\rm earth}}} = 11.2 \text{ km s}^{-1} = 3.73 \times 10^{-5} c, \qquad \gamma = 1 + 6.95 \times 10^{-10} \simeq 1$$

(3) The centrifugal barrier is not present in the effective potential for $\tilde{L} < \sqrt{12}R_g$, and this condition translates to

$$b < b_{\mathrm{max}} := \frac{\sqrt{12}R_g}{\gamma v} = 5.07 \text{ m}$$

Objects falling toward the Moon with this impact parameter will surely fall into the black hole, solving the toxic waste problem but not the energy one, because no energy is extracted in this case.

This impact parameter b_{max} is reported in the table above, and subtends 2.77×10^{-3} arcsec. It depends on $(\gamma v)^{-1}$, so a slower object will fall more easily into the black hole; indeed an object starting at rest ($v = 0$) will surely fall radially into the black hole ($b_{\mathrm{max}} \to \infty$), you need some speed to pass through without falling, the faster the better.

(4) The computation of the orbit pericenter proceeds as in the computation of the photon radius, but the object here has a mass. We need to set to zero the quantity

$$\frac{dr}{d\varphi} = \frac{dr}{d\tau}\frac{d\tau}{d\varphi}$$

however the (inverse of the) second term

$$d\varphi/d\tau = p^{\varphi}/m = \gamma v b/r^2$$

is purely multiplicative, so the pericenter will be obtained by the condition $dr/d\tau = 0$. After some math, recalling that $\gamma^2 = 1/(1 - v^2)$, I obtain:

$$\left(\frac{dr}{d\tau}\right)^2 = \gamma^2 \left[1 - \left(1 - \frac{2R_g}{r}\right)\left(1 + \frac{v^2}{r^2}(b^2 - r^2)\right)\right] = 0$$

that can be trasformed to:

$$2R_g r_{\mathrm{peri}}^2 - v^2(b^2 - r_{\mathrm{peri}}^2)(r_{\mathrm{peri}} - 2R_g) = 0$$

If the packet is required to have its pericenter at a radius $r_{\mathrm{peri}} > R_{\mathrm{isco}}$, so as to crash onto an existing hot accretion disc and be thermalized, then $b > b_{\mathrm{min}}$, where b_{min} is obtained by setting $r_{\mathrm{peri}} = r_{\mathrm{isco}} = 6R_g$ in the equation:

$$b_{\mathrm{min}} = 6R_g\sqrt{1 + \frac{1}{2v^2}} = 6.21 \text{ m}$$

This length subtends 3.39×10^{-3} arcsec.

(5) There is no unique answer to this question. We want the packets not to fall directly to the black hole Moon, otherwise there would be no energy gain. We want the material to rotate around the black hole many times, so we will launch the packets at many gravitational radii, so as to merge with the accretion disc and heat up. For such a tiny black hole we can rely on tides to destroy the object when he gets near the accretion disc. The orbit will be either parabolic or highly elliptical, we can assume that

a circular orbit will be gained by viscous interaction with the disc, and this poses the problem on how to create the disc as a start.

The Eddington rate (the rate at which accretion is limited by radiation pressure) of such a black hole is 58000 ton/s, so the black hole should accrete at sub-Eddington rates unless we are capable of sending a huge continuous flux of matter. Its Eddington luminosity is 5.22×10^{30} erg/s, below the solar luminosity but not by much, another good reason to limit accretion (the cost of geo-engineering to compensate for the extra energy and avoid runaway greenhouse effect may be too high). However, accretion much below the Eddington rate is not radiatively efficient, so we cannot go too low.

The main problem is how to collect the radiated energy: we could think of a huge array of solar panels that surround the black hole moon and then send the energy to earth via radio impulses. But why not doing the same thing with the Sun? collecting energy without shadowing the Earth would be way more efficient and safe.