Cosmology 1

2025/2026
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First intermediate test
Topic: general relativity.
Deadline: 20 March, 9:00.

An extended body that crosses the event horizon is subject to a finite tidal force, whose strength scales with the black hole mass. As a consequence, while entering a stellar-mass black hole is fatal to a human, little harm is suffered when crossing the horizon of a supermassive black hole.

The equation that represents tides is the geodesic deviation equation. Take two nearby geodesics, one passing through an event P and flowing along a vector V^μ, and another one passing through a nearby event, and call ξ^ν the vector that joins the two geodesics. The stress that this vector suffers is determined by:

$$\nabla_V \nabla_V \xi^\alpha = R^\alpha_{\ \mu\nu\beta} V^\mu V^\nu \xi^\beta$$

where the operator $\nabla_V A^\nu = V^\mu A^\nu_{;\mu}$ (here applied to a generic vector A^ν) is the covariant derivative along the vector V^μ, and $R^\alpha_{\ \mu\nu\beta}$ is the Riemann tensor.

(1) To keep math simple, we assume that a human body is momentarily still at the event $(0, r, \pi/2, 0)$, so that its timeline has only a component along the time component. Call ξ^μ the space-like vector that describes its length, from head to feet, and assume that it has only a component along radius r. Write the geodesic deviation equation in this simple case, discussing how you simplified the left-hand term and how you computed the (only) component of the Riemann tensor you need.

(2) Repeat the computation assuming that the body stretches along φ component.

(3) Now interpret the equation, and argue that it represents the tidal force (per unit mass) that the body feels. How would you define a "spaghettification" criterion at the event horizon? express a condition on black hole mass (in solar masses) that tells what black holes do not kill the human when he/she crosses the event horizon.

(4) Identify all the approximations made to get to this results, and quickly discuss what is required to get to a more realistic calculation.

But please remember, the result must be shown in not more than four pages. Give the main analytic results but do not report all the steps.

Solution

(1) Using the condition $V^\mu V_\mu = -1$, and assuming that this vector has vanishing spatial components, we obtain:

$$V^\mu = \left(\frac{1}{\sqrt{1 - 2GM/r}}, 0, 0, 0 \right)$$

Let's first compute second covariant derivative $\nabla_V \nabla_V \xi^\alpha$, where $\xi^\alpha = (0, \ell, \pi/2, 0)$ and ℓ is the length of the human body. The geodesic deviation equation is second order, so to solve it we must fix the value of ξ^α and of its first (covariant) derivative. If the two geodesics are parallel, then we must have that $\nabla_V \xi^\alpha = 0$. On the other hand we have, for the r component:

$$\nabla_V \xi^r = V^\mu \xi^r_{;\mu} = V^\mu \xi^r_{,\mu} + V^0 \Gamma^r_{r0} \xi^r = \frac{d\xi^r}{d\tau} = 0$$

where the last step is motivated by the fact that $\Gamma^r_{r0} = 0$. It follows that:

$$\nabla_V \nabla_V \xi^r = V^\nu \left(\nabla_V \xi^r \right)_{;\nu} = V^\nu \left(\frac{d\xi^r}{d\tau} \right)_{,\nu} = \frac{d^2 \xi^r}{d\tau^2}$$

The right-hand side of the r-component of the geodesic deviation equation reduces to:

$$R^r_{\mu\nu\beta} V^\mu V^\nu \xi^\beta = R^r_{00r} \xi^r \frac{1}{1 - \frac{2GM}{r}}$$

In Schwartzschild metric we have that:

$$R^r_{00r} = \frac{2GM}{r^3} \left(1 - \frac{2GM}{r} \right)$$

This can be obtained from the $R^0_{r0r} = 2GM/r^2(r - 2GM)$ component, that is usually computed, by lowering the first index, then inverting the first two indices while changing sign, then raising the index again. The geodesic deviation equation then gives:

$$\frac{d^2 \ell}{d\tau^2} = \frac{2GM}{r^3} \ell$$

The right-hand term of the geodesic deviation equation vanishes for all other components, because it implies components of the Riemann tensor like $R^\alpha_{00\beta}$, with $\alpha \neq \beta$, so the acceleration is non-vanishing only for the radial component, and this reinforces our interpretation that ξ^α remains radial.

(2) A very similar reasoning applies when the body stretches along the φ direction. In this case $\xi^\alpha = (0, 0, \pi/2, \ell/r)$ (the ϕ component is an angle), so:

$$\frac{d^2 \ell}{d\tau^2} = R^\varphi_{00\varphi} \ell \frac{1}{1 - \frac{2GM}{r}} = -\frac{GM}{r^3} \ell$$

(3) In these equations the left term is an acceleration, the right term is proportional to the extension of the body, so it gives the tidal force. This force does not diverge at the event horizon, so the tides are finite when crossing the horizon and do not necessarily kill our human. (It must be noticed that the possibility of having a trajectory that is momentarily still is impossible when we approach the event horizon.)

Tides are positive in the radial direction and negative in the tangential one, so the body is stretched radially and compressed tangentially, hence the name "spaghettification". As we are adapted to live with a gravitational acceleration g, we set a threshold criterion on the acceleration caused by tides as:

$$\frac{GM_{\rm thr}}{R_s^3}\ell = g$$

We can make c explicit here by setting $R_s = 2GM/c^2$. This threshold translates to a threshold in mass:

$$M_{\rm thr} = \frac{c^3}{G}\sqrt{\frac{\ell}{8g}}$$

Adopting $g = 9.81$ m s^{-2} and $\ell = 1.80$ m, and expressing the mass in solar masses, I obtain:

$$M_{\rm thr} = 3.07 \times 10^4 \ {\rm M}_\odot$$

(4) The two main approximations that are done here are the fact that the black hole is not rotating and that the trajectory of the human body is irrealistic. Staying within the Schwartzschild metric, it would be much more realistic to adopt for the V^μ vector a realistic trajectory that may be closed or even open. This would imply a substantial increase in the amount of analytic calculations.