Cosmology 1

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Proposed problem, lecture 6
Topic: curved universes.

Spherical coordinates in a 4D flat space can be generalized as:

$$
\begin{aligned}
x^1 &= R \sin\chi \sin\vartheta \sin\varphi \\
x^2 &= R \sin\chi \sin\vartheta \cos\varphi \\
x^3 &= R \sin\chi \cos\vartheta \\
x^4 &= R \cos\chi
\end{aligned}
$$

where $R^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2$. Follow the procedure outlined in the Vittorio textbook on page 8 to obtain the metric of a 3-sphere. Alternatively, find the coordinate transformation matrix $\partial\bar{x}^i/\partial x^j$ and use it to work out the basis vectors of this 4D space in spherical coordinates from those in Cartesian coordinates, then recall that the metric can be expressed as the vector products of the basis vectors, $g_{ij} = \vec{e}_i \cdot \vec{e}_j$. This technique can be used to demonstrate that the metric in the $(R, \chi, \vartheta, \varphi)$ takes the form:

$$ds^2 = dR^2 + R^2 \left[d\chi^2 + sin^2\chi(d\vartheta^2 + \sin^2\vartheta d\varphi^2) \right]$$

From this work out the metric of a 3-sphere. How does it relate to the metric of a closed universe?

Use your favourite plotting program to plot, for $k = 1$,0 and -1, the length of a circle at coordinate r, the area of a surface at coordinate r and the volume within r as a function of the χ variable.