Cosmology 1

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Second intermediate test
Topic: FRW models.

Suppose the luminosity distance is accurately measured using type Ia SNe, with these results at five redshifts:

z	d_L (Mpc)
0.1	474.8 ± 9.3
0.5	2911 ± 58
0.8	5213 ± 92
1.0	6610 ± 150
1.5	11340 ± 240

(a) Assuming a flat ΛCDM model ($\Omega_\Lambda > 0$, $\Omega_m + \Omega_\Lambda = 1$, no radiation or curvature), estimate the cosmological parameters H_0 and Ω_m (and their uncertainty) from these measurements.

(b) Would an open Universe (without Λ) or a de Sitter Universe fit these data?

(c) For the best-fit parameters that you obtained, plot the scale factor versus time as in the proposed exercise of lecture 17, adding a Milne model with the same Hubble constant. Report in the plot, in some way you find convenient, the uncertainty on the age of the universe. Would a globular cluster of age 13.1 ± 0.5 Gyr be a problem for this model?

(d) Now the more difficult part: fix H_0 to the best-fit value you obtained, and do not restrict the analysis to flat universes. What region of the $\Omega_m - \Omega_\Lambda$ parameter space would be consistent with the above measurements? try to get to the final answer, or describe a roadmap to obtain the result and list the difficulties that you have met in the process. If you get an answer, take a look at Fig. 2.11 of Vittorio textbook, and give a brief discussion of the result.

Note: I constructed this exercise starting form Planck-like parameter values (see the notes of Lecture 18), so you should obtain comparable results. However, due to the known tension on the Hubble constant, type Ia SNe would give a higher value of H_0.

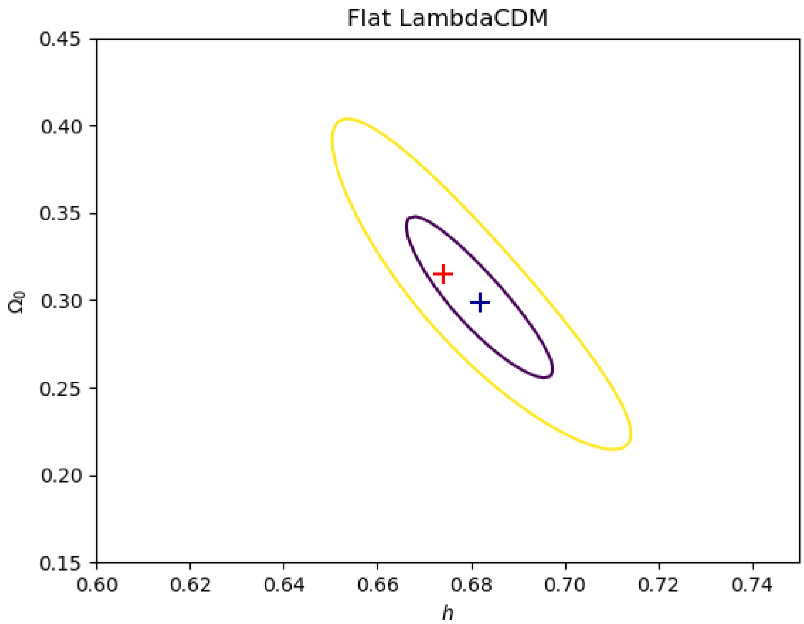

Figure 1:

Solution

The provided dataset was produced by computing the luminosity distance for a flat model with $\Omega_0 = 0.315$ and $H_0 = 67.4$ km s^{-1} Mpc^{-1}. The observed luminosity distance d_{Lo} at the five (arbitrarily chosen) redshifts was then computed as a Gaussian distributed variable with mean equal to the theoretical value d_{Lt} and standard deviation equal to 2% of the mean. Errors σ_o were computed as the d_{Lt} times a Gaussian variable with mean 0.02 and standard deviation 0.002, to reproduce (in a qualitative way) the uncertainty on the standard deviation estimated from a data set. The $\chi^2 = \Sigma_i (d_{Lo,i} - d_{Lt})^2/\sigma^2_{o,i}$ of this distribution results 2.38.

This test requires numerical techniques for the computation of the integral for the luminosity distance, and for the solution of the Friedmann equation for non-flat cases. I have solved the problem using python, with numpy and scipy libraries and matplotlib for graphics. Because the data reach $z = 1.5$, using the phenomenological expression for d_L based on the Taylor expansion in redshift, that contains the acceleration parameter q_0, would not give an accurate answer.

(a) In the flat model the expression of the scale factor is analytic, but the luminosity distance must be numerically computed as:

$$d_z(z) = (1+z) \int_{t(z)}^{t_0} \frac{cdt}{a(t)}$$

The result depends on the two parameters H_0 and Ω_m. In this simple case it is possible to just sample the 2D parameter space in a regular grid. After some iterations to fix the investigated region of parameter space, I used

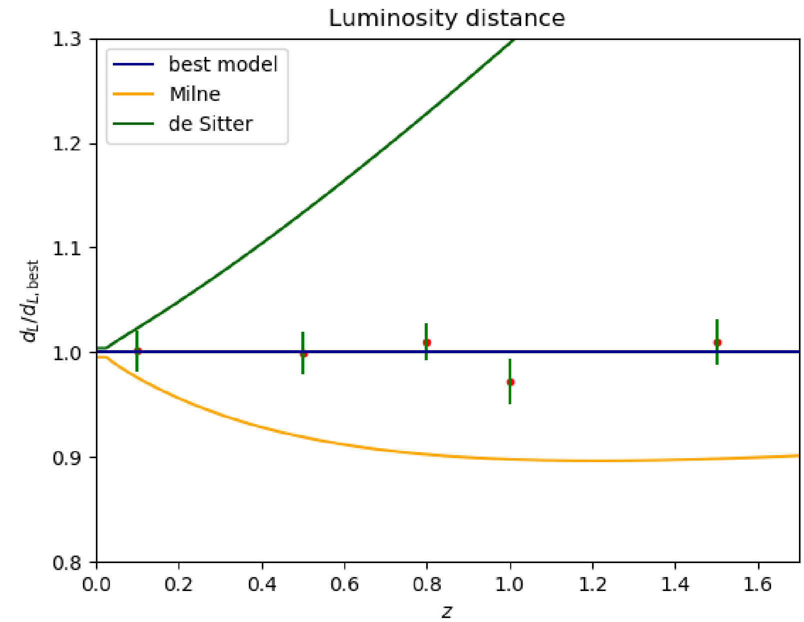

Figure 2:

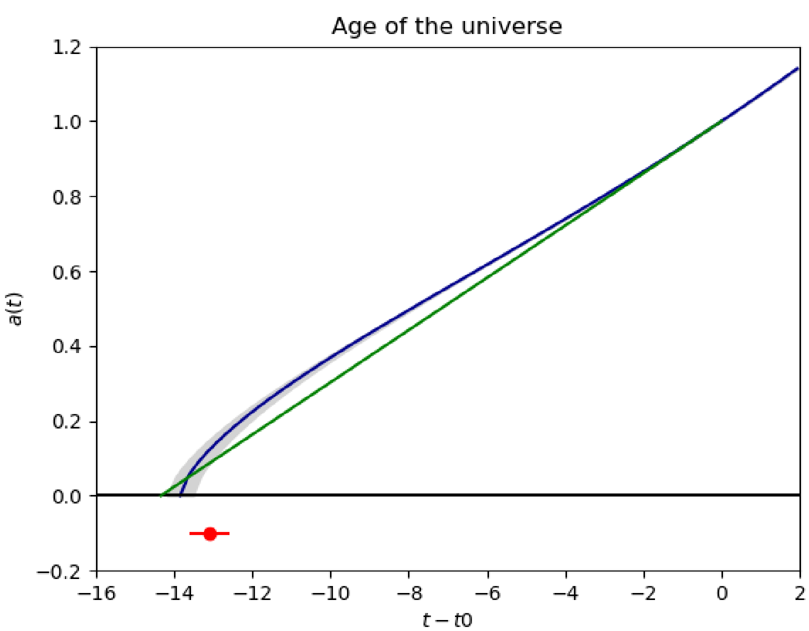

Figure 3:

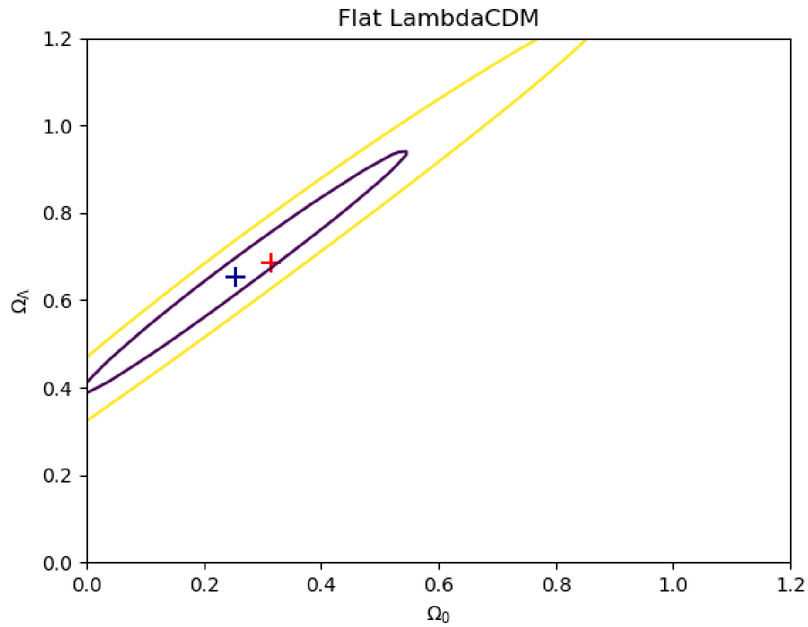

Figure 4:

100×100 parameter values in the range $h \in [0.60, 0.75]$, $\Omega_0 \in [0.15, 0.45]$. I computed d_{Lt} for each parameter combination and compared it with the data using the χ^2 statistics defined above. I quantify this information through the 68% and 95% confidence levels, that for 5 datapoints and 2 parameters (3 degrees of freedom) are given by $\chi^2_{68} = 3.36$ and $\chi^2_{95} = 7.81$. Figure 1 gives the result, in terms of contour plot of the $\chi^2(H_0, \Omega_0)$ function at the two levels defined above. These are the confidence ellipses. Their projection on the axes give the errorbars roughly corresponding to 1-σ and 2-σ. The orientation of the ellipses gives the degeneracy of the two parameters: changes along the ellipses major axis will give little difference on the goodness of fit. The resulting values are:

$$h = 0.68 \pm 0.02$$

$$\Omega_0 = 0.30 \pm 0.04$$

(1-σ errors). The figure reports also the true value used to generate the data (red cross) and the best fit value (blue cross).

(b) A complete solution of the second question can be given after having fully solved the fourth point. However, we know that the Hubble constant is mostly constrained by the low-z data, while higher-z data constrain the acceleration parameter, so we can have an indication of the consistency of these cosmologies by plotting the luminosity distance for a Milne model ($\Omega_m = \Omega_\Lambda = 0$, $\Omega_k = 1$) and a de Sitter model ($\Omega_\Lambda = 1$, $\Omega_m = \Omega_k = 0$) using the same Hubble constant as its obtained best value. Luminosity distances can be easily obtained analytically in this case. Figure 2 gives the results, here we plot the ratio of d_L with respect to that of the best model, otherwise errorbars and differences would be hard to see. Both

Milne and de Sitter models are clearly inconsistent with evidence at high redshift; it is easy to check that changing the Hubble constant does not help, and that $\Omega_m > 0$ in the open model does not help either.

(c) Figure 3 shows the resulting scale factor as a function of time. During the calculation of the χ^2 I stored, for each t, the smallest and the largest $a(t)$ values for models that have $\chi^2 \leq \chi^2_{68}$. These are reported in the plot as the shaded area. The width of this area at $a = 0$ gives the uncertainty in the age of the Universe:

$$t_0 = 13.9 \pm 0.3$$

The plot reports also the age of the globular cluster, that is younger than the age of the Universe with a statistical significance of a little more than 1σ. This means that there is no strong evidence of an age problem, but there are models compatible with the luminosity distance data that would be disfavoured by the measured globular cluster age. One could conclude that adding the constraint from the age of the globular cluster could reduce the uncertainty on the parameters.

(d) To obtain parameter constraints in the $\Omega_m - \Omega_\Lambda$ plane one has to redo the calculation performed for point (a), with very similar technical details. However, there are two difficulties. First, the solution of the Friedmann equation cannot be obtained analytically, so one has to resort to numerical integration. Second, the universe is not flat here, so the luminosity distance must be computed as follows:

$$d_c = \int_t^{t_0} \frac{c dt}{a(t)}$$

$$\mathcal{R}_0 = \frac{c}{H_0\sqrt{|\Omega_k}}$$

$$d_L = \frac{1}{a}\mathcal{R}_0 \sin\left(\frac{d_c}{\mathcal{R}_0}\right) \quad \text{if} \quad \Omega_k < 0$$

$$d_L = \frac{1}{a}\mathcal{R}_0 \sinh\left(\frac{d_c}{\mathcal{R}_0}\right) \quad \text{if} \quad \Omega_k > 0$$

In this case I used 100×100 parameter values in the range $\Omega_m \in [0, 1.2]$ and $\Omega_\Lambda \in [0, 1.2]$. The results are shown in Figure 4: due to the strong degeneracy in the two parameters, the data now are compatible with a much larger set of model parameters, and this shows how the starting hypothesis ("prior") on the parameter space can change parameter estimation. Yet, we can confirm for this plot that the Milne universe and the de Sitter universe are incompatible with data.

As a conclusion, the measurement of luminosity distance can tightly constrain parameters if the Universe is flat, but has low constraining power on the curvature. Besides, the position of the acoustic peak of the CMB is able to strongly constrain the curvature, so the joint use of CMB and SNe data is the key to achieve accuracy.

As a matter of fact, CMB alone is able to give very good constraints to cosmological parameters, while SNe data give incompatible values of H_0. The meaning of this "tension" is still under investigation.