Cosmology 1

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Prof. Pierluigi Monaco

Solution of proposed problem, lecture 1
Topic: special relativity, see also exercise 1.19 of Schutz textbook.

(a) Assuming that in the Galaxy frame the spaceship moves at speed v along the x axis, the velocity and acceleration vectors in the MCRF are $\vec{U} \to (1, 0, 0, 0)$, $\vec{A} \to (0, a, 0, 0)$. To find the components of $\vec{U}$ and $\vec{A}$ in the Galaxy reference frame, let's apply an inverse Lorentz transformation, obtaining: $\vec{U} \to (\gamma, \gamma v, 0, 0)$, $\vec{A} \to (\gamma a v, \gamma a, 0, 0)$.

The equations of motion are:

$$\frac{d\vec{U}}{d\tau} = \vec{A}$$

The 0 and 1 components give, respectively:

$$\frac{d\gamma}{d\tau} = \gamma a v, \quad \frac{d\gamma v}{d\tau} = \gamma a$$

Combining together the two equations, one obtains:

$$\frac{dv}{d\tau} = a(1 - v^2)$$

whose solution is

$$v(\tau) = \tanh(a\tau)$$

The Lorentz γ factor results:

$$\gamma(\tau) = \frac{1}{\sqrt{1 - v^2}} = \cosh(a\tau)$$

(b) To compute the traveled distance ℓ during the first half of the travel, it is sufficient to recall that

$$\frac{d\vec{x}}{d\tau} = \vec{U}$$

Assuming that in the Galaxy frame $\vec{x} \to (t, \ell, 0, 0)$, the 0 and 1 components of this equation give:

$$\frac{dt}{d\tau} = \gamma = \cosh(a\tau), \quad \frac{d\ell}{d\tau} = \gamma v = \sinh(a\tau)$$

that are easily integrated to obtain:

$$t(\tau) = \frac{1}{a}\sinh(a\tau)$$

$$\ell(\tau) = \frac{1}{a}(\cosh(a\tau) - 1)$$

Now, calling $g = 9.8\ m\,s^{-2}$, a for $c = 1$ gets the value of $a = g/c\ s^{-1} = 3.3 \times 10^{-8}\ s^{-1} \sim 1.0\ \mathrm{yr}^{-1}$ This means that the spaceship will reach relativistic speed after ~ 1 yr of constant acceleration. It is convenient to recast the distance to cover (4000 pc) in light years, $\ell = 13000$ yr, so as to obtain that

$$\cos(a\tau) - 1 \sim 13000$$

The -1 term will clearly be negligible in this case, so that it is easy to get the proper time at half of the Galaxy: $a\tau \sim 10$, $\tau \sim 10$ yr. The spaceship will thus get to the Galaxy center in 20 yr, in proper time..

We now notice that when $a\tau \gg 1$ then $\tanh(a\tau) \sim 1$ and $\sinh(a\tau) \sim \cosh(a\tau)$, so $v \to 1$ and $t \sim \ell$. So the Galaxy-frame time t to get to a distance ℓ is very similar to that computed as if the spaceship where going at the speed of light all the time: $t \sim 26000$ yr.

(c) For large $a\tau$ values it is easy to get that $\ell a \sim \cosh(a\tau) = \gamma$, so that the maximum γ factor of the spaceship is 13000. This is a completely unrealistic, ultra-relativistic value.

(d) The space-time diagram is easily constructed with a tool like python, and will show a time-like trajectory that, on time scales of ~ 1 yr will asymptote to a null trajectory. At the maximum speed the axes of the MCRF will be almost collapsed to the bisector null line.