Cosmology 1

2023/2024
Prof. Pierluigi Monaco

Solution of proposed problem, lecture 18
Topic: inflation.

It is useful to express all results as a function of the starting time of inflation t_i and the number of e-folds N_e. The evolution of the scale factor follows the three phases:

1. $a(t) = a_i(t/t_i)^{1/2}$, where a_i is the scale factor at the start of inflation and t_i is given. In this phase the Hubble parameter is $H = 1/2t$, so at the start of inflation $H_i = 1/2t_i$. The first derivative of the scale factor is:

$$\dot{a} = \frac{a_i}{2t_i}\left(\frac{t}{t_i}\right)^{-1/2}$$

2. The scale factor evolves exponentially from t_i to t_f. H is constant in this phase, so we put it equal to H_i to guarantee continuity of the scale factor and its first derivative. So we have:

$$a(t) = a_i \mathrm{e}^{H_i(t-t_i)}$$

At $t = t_f$ the scale factor is $a_f = a_i \exp N_e$. Moreover,

$$t_f = N_e H_i^{-1} + t_i = (2N_e + 1)t_i$$

3. We must notice here that an extrapolation of the scale factor function of this radiative phase at $t < t_f$ will *not* reach $a = 0$ at $t = 0$, otherwise the universe would be radiative at all times. So we assume that

$$a(t) = a_f \left(\frac{t - t_3}{t_f - t_3}\right)^{1/2}$$

where t_3 it the hypothetical Big Bang time obtained by extrapolating this phase to earlier times. The scale factor is thus continuos at t_3, we must also force continuity of the Hubble parameter: $H(t_f) = 1/2(t_f - t_3) = H_i$. We then obtain that:

$$t_3 = t_f - \frac{1}{2H_i} = 2N_e t_i\,.$$

As a consequence:

$$a(t) = a_i \mathrm{e}^{N_e}\left(\frac{t}{t_i} - 2N_e\right)^{1/2}\,.$$

Asking then that $a(t_0) = 1$, and considering that $t_0/t_i = 4.36 \times 10^{52} \gg 2N_e$, we get:

$$a_i = \mathrm{e}^{-N_e} \left(\frac{t_i}{t_0} \right)^{1/2} = 4.79 \times 10^{-27} \mathrm{e}^{-N_e} ,$$

$$a_f = \left(\frac{t_i}{t_0} \right)^{1/2} = 4.79 \times 10^{-27} .$$

The comoving horizon evolves like this:

$$d_{cH} = \frac{2ct_i}{a_i} \left(\frac{t}{t_i} \right)^{1/2} \qquad t < t_i$$

$$d_{cH} = \frac{2ct_i}{a_i} \mathrm{e}^{-H_i(t-t_i)} \qquad t_i < t < t_f$$

$$d_{cH} = \frac{2ct_i}{a_i} \mathrm{e}^{-N_e} \left(\frac{t}{t_i} - 2N_e \right)^{1/2} \qquad t_f < t < t_0$$

The condition $d_{cH}(t_i) > d_{cH}(t_0)$ then becomes $\exp(-N_e)(t_0/t_i)^{1/2} < 1$, or:

$$N_e > \frac{1}{2} \ln \frac{t_0}{t_i} = 60.6 .$$

The conformal time

$$\eta = \int_0^{t_0} \frac{cdt}{a(t)}$$

is decomposed into three integrals over the three expansion phases: $\eta = \Delta\eta_1 + \Delta\eta_2 + \Delta\eta_3$. Using the solutions found above:

$$\Delta\eta_1 = \frac{2ct_i}{a_i}$$

$$\Delta\eta_2 = \frac{2ct_i}{a_i} \left(1 - \mathrm{e}^{-N_e} \right) \simeq \frac{2ct_i}{a_i}$$

$$\Delta\eta_3 = \frac{2ct_i}{a_i} \mathrm{e}^{-N_e} \left[\left(\frac{t_0}{t_i} - 2N_e \right)^{1/2} - 1 \right] \simeq \frac{2ct_i}{a_i} \mathrm{e}^{-N_e} \left(\frac{t_0}{t_i} \right)^{1/2}$$