Cosmology 1

2023/2024
Prof. Pierluigi Monaco

Solution of proposed problem, lectures 2 and 3
Topic: The spaceship gets to Sgr A*.

(a) The Schwartzschild radius of a $4 \times 10^6\ M_\odot$ black hole is equal to (we explicit the speed of light for this calculation) $R_s = 2GM/c^2 = 1.18 \times 10^{10}$ m $= 0.079$ astronomical units (AU), so $r_0 = 1000R_s$ correspond to 79 AU.

(b) Calling $r_{\rm pr}$ the probe's radial coordinate, for a radial trajectory the φ coordinate does not change so $\tilde{L} = 0$, while $\tilde{E}$ can be fixed by imposing that $dr_{\rm pr}/d\tau = 0$ at $r_{\rm pr} = r_0 = 1000R_s$ at which the probe is released: $\tilde{E}^2 = 1 - R_s/r_0 = 1 - 1/1000 = 0.999$. The equation of motion can then be written as:

$$\left(\frac{dr_{\rm pr}}{d\tau}\right)^2 = \tilde{E}^1 - 1 + \frac{2GM}{r} = -\frac{2GM}{r_0} + \frac{2GM}{r}$$

(c) Gravitational redshift is defined as $1 + z = \nu_{em}/\nu_{obs}$. The signal is lost when $\nu_{obs} = \nu_{em}/10$, for which $z = 9$. If the probe radial motion is neglected, spaceship (ss) and probe (pr) have four-velocities with time components $U^0_{\rm ss} = 1/\sqrt{1 - 2GM/r_0}$ and $U^0_{\rm pr} = 1/\sqrt{1 - 2GM/r_{\rm pr}}$, and vanishing space components. Calling $\vec{f}$ the photon's momentum, the photon energy, as seen in the two frames, is respectively $E_{\rm obs} = -f_0 U^0_{\rm ss}$ and $E_{\rm em} = -f_0 U^0_{\rm pr}$, with f_0 being constant along the photon geodesic. Then $\nu_{obs}/\nu_{em} = E_{\rm obs}/E_{\rm em} = 10$ gives:

$$r_{\rm lost} = 1.01009 R_s$$