Cosmology 1

2018/2019
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Third intermediate test
Topic: early universe.

Problem 1. This is an extended version of the proposed problem of lecture 21.

Consider a flat model with $H_0 = 67.4$ km s^{-1}, and ignore the contribution of Ω_Λ. Starting from the Big Bang, the scale factor evolves as follows: (1) from $t = 0$ to $t = t_i = 10^{-35}$ s it evolves like a radiation-dominated universe, (2) at t_i an exponential (de Sitter) inflationary phase starts with Hubble parameter H_i, lasting for N_e e-folds up to $t = t_f$ (so that $H(t_f - t_i) = N_e$), (3) from t_f to $t_{\rm eq}$, the time of equivalence, it evolves again like a radiation-dominated universe, (4) from $t_{\rm eq}$ to the present time t_0 it evolves like a matter dominated universe. We know that the equivalence redshift is $z_{\rm eq} = 3400$, and compute t_0 from the Hubble constant H_0 for a matter-dominated universe (it will not be correct, but it is a small difference for our purposes). In this evolution the scale factor $a(t)$ and the Hubble parameter $H(t)$ remain continuous.

1. Find a (piecewise) analytic description of the scale factor $a(t)$ for the whole evolution of this universe.

2. Compute the dimension of the Hubble comoving horizon $d_{cH} = c/\dot{a}$ (in units of Mpc) at $t = t_i$ and at $t = t_0$, and find the minimum number $N_{e,min}$ of e-folds needed to (barely) solve the horizon problem.

3. Assume now that inflation lasted $N_e = N_{e,min} + 1$ e-foldings. Find a way to visualize in a (rigorous) plot the scale factor and the comoving Hubble horizon for the whole time span of the universe. Report the times that limit the four cosmic eras in all plots. In the plot of the Hubble comoving horizon, report a scale that is seen on the CMB outside the horizon at recombination (justify the choice in a quantitative way).

4. For the same case, can you draw an accurate conformal diagram of this universe, marking the position of the visible horizon (at $z = 1100$)?

5. Compute the comoving size of the particle horizon and of the visible horizon, in Mpc, for this model, and compare them.

6. Qualitatively, what would happen to these plots if inflation lasted 10 more e-folds?

Note: recall that, for the second radiative era and the matter-dominated era, the extrapolation of the scale factor back in time, before the starting time of the era itself, **does not** *go to 0 for $t = 0$. So the introduction of a new era implies a new time parameter, as in the solution of Proposed problem 21.*

Problem 2.

Neutrino decoupling, that causes the freezing of the neutron-to-proton ratio n_n/n_p, takes place when the timescale for weak interactions τ_w is equal to the cosmic time t. Cosmic time, at fixed temperature T, depends on the number of particles through $g^\star$. Suppose that the number of neutrino types is N_ν, not necessarily 3; using the approach given in the lecture notes, calculate how the temperature at decoupling $T_{\nu,\rm dec}$ depends on N_ν.

Note: the density at the denominator of τ is supposed to be the density of electron pairs. You will not obtain for $N_\nu = 3$ the 900 keV value quoted in the notes and in Bonometto textbook; feel free to rescale the number you obtain to 900 keV, we are concerned on relative differences as a function of N_ν.

Use this information to compute the dependency of n_n/n_p on N_ν, and then estimate how much the *He* abundance would change if we had 2 or 4 neutrinos. Can you identify the approximated steps one should work on to obtain more reliable numbers?

Solution of problem 1

Please read first the solution of the proposed problem 21.

1. The scale factor evolves in four phases:

$$a_1(t) = a_i \left(\frac{t}{t_i}\right)^{1/2}$$

$$a_2(t) = a_i \mathrm{e}^{H(t-t_i)}$$

$$a_3(t) = a_f \left(\frac{t - t_3}{t_f - t_3}\right)^{1/2}$$

$$a_4(t) = a_{\mathrm{eq}} \left(\frac{t - t_4}{t_{\mathrm{eq}} - t_4}\right)^{2/3}$$

Here we have supposed that the extrapolations of $a(t)$ in phases 3 and 4 go to zero at times t_3 and t_4. The known data are H_0, t_i, $a_{\mathrm{eq}} = 1/(1+z_{\mathrm{eq}})$ and of course $a_0 = a(t_0) = 1$. We require that the four functions $a(t)$ and $H(t) = \dot{a}/a$ are continuous at times t_i, t_f and t_{eq}, we set $a(t_0) = 0$ and $H(t_0) = H_0$, and we exploit the fact that $t_{\mathrm{eq}} \gg t_f$ and $t_0 \gg t_{\mathrm{eq}}$. We then obtain, as a function of a_i and N_e:

$$a_f = a_i \mathrm{e}^{N} e$$

$$t_f = (2N_e + 1)t_i$$

$$t_3 = 2N_e t_i$$

$$t_{\mathrm{eq}} = t_i \left(\frac{a_{\mathrm{eq}}}{a_i \mathrm{e}^{N_e}}\right)^2$$

$$t_4 = -\frac{1}{3} t_{\mathrm{eq}}$$

$$t_0 = \frac{2}{3H_0}$$

The (known) value for a_{eq} is obtained from $a_3(t_{\mathrm{eq}})$ and $a_4(t_{\mathrm{eq}})$ (assuming $a_4(t_0) = 1$), and this allows to find a_i:

$$a_i = \mathrm{e}^{-N_e} a_{\mathrm{eq}}^{1/4} \left(\frac{4t_i}{3t_0}\right)^{1/2}$$

2. The comoving Hubble horizon for the four phases is:

$$d_{\mathrm{cH1}} = \frac{2ct_i}{a_i} \left(\frac{t}{t_i}\right)^{1/2}$$

$$d_{\mathrm{cH2}} = \frac{2ct_i}{a_i} \mathrm{e}^{-H(t-t_i)}$$

$$d_{\mathrm{cH3}} = \frac{2ct_i}{a_i} \mathrm{e}^{-N_e} \left(\frac{a(t)}{a_f}\right)$$

$$d_{\mathrm{cH4}} = \frac{2ct_i}{a_i}\mathrm{e}^{-N_e}\left(\frac{a_{\mathrm{eq}}}{a_f}\right)\left(\frac{a(t)}{a_{\mathrm{eq}}}\right)^{1/2}$$

As discussed in the lecture notes, the condition $d_{cH}(t = t_0) = d_{cHi} = 2ct_i/a_i$ gives:

$$\mathrm{e}^{N_e} = \frac{a_{\mathrm{eq}}}{a_f}\left(\frac{1}{a_{\mathrm{eq}}}\right)^{1/2}$$

Using for a_f the expression given above, one obtains:

$$N_{e,min} = \frac{1}{4}\ln a_{\mathrm{eq}} + \frac{1}{2}\ln\frac{3t_0}{4t_i} = 58.2$$

For this N_e the size of the comoving Hubble horizon at the end of the four phases is: $d_{cH1} = 4450$ Mpc, $d_{cH2} = 2.24 \times 10^{-22}$ Mpc, $d_{cH3} = 76$ Mpc, $d_{cH0} = 4450$ Mpc.

3. For $N_e = N_{e,min} + 1$ the size of the comoving Hubble horizon at the end of the beginning of inflation becomes $d_{cH1} = 12100$ Mpc, the other values remain unchanged.

   The plots are shown in the first figure; in this case a log-log plot is clearly necessary. Here the recombination time t_{rec} is obtained by computing the time at which $a_4(t_{\mathrm{rec}}) = a_{\mathrm{rec}} = 1/(1 + z_{\mathrm{rec}})$; in this case we do not assume that $t_{\mathrm{rec}} \gg t_{\mathrm{eq}}$, obtaining $t_{\mathrm{rec}} = 7.98 \times 10^{12}$ s $= 2.5 \times 10^5$ yr. The horizon at the equivalence is 134 Mpc, we report in the plot a scale 10 times larger, still below the horizon value today.

4. Figure 2 shows the conformal diagram in this case. For $N_e = N_{e,min} + 1$ recombination is clearly visible. The contributions to the conformal time η are:

$$\Delta\eta_1 = \frac{2ct_i}{a_i} = 12100 \text{ Mpc}$$

$$\Delta\eta_2 = \frac{2ct_i}{a_i}(1 - \mathrm{e}^{-N_e} \simeq \frac{2ct_i}{a_i} = 12100 \text{ Mpc}$$

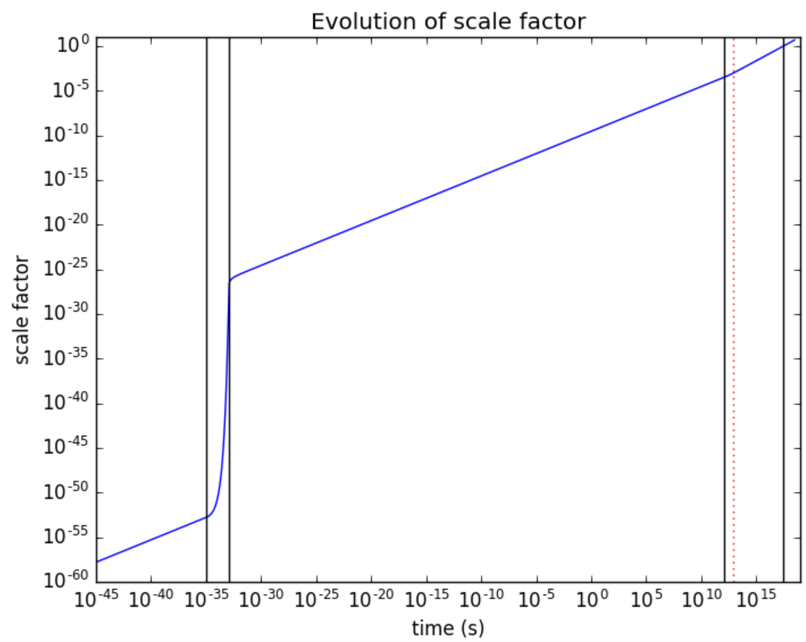
$$\Delta\eta_3 = \frac{2ct_i}{a_i}\mathrm{e}^{-N_e}\left(\frac{t}{t_i}\right)^{1/2} = 76 \text{ Mpc}$$

$$\Delta\eta_4 \simeq 3ct_0 = 8900 \text{ Mpc}$$

   The third contribution is negligible, but $3ct_0 = 8900$ Mpc is not very much smaller than the first and second contributions, given that the horizon problem is barely solved. The visible horizon is, within the resolution of the plot, at $3ct_0$.

5. The comoving particle horizon can be simply computed by summing all the contributions to η. It results of 33200 Mpc, larger than $3ct_0$ but only by a factor of order unity.

6. If we add 10 more e-foldings, the particle horizon at t_0 results of 5.3×10^8 Mpc, while the comoving Hubble horizon at the beginning of inflation is 2.6×10^8 Mpc. The plots of the scale factor and comoving Hubble horizon can be done by further extending the already wide range of values in the y-axis, but the visible horizon disappears from the conformal diagram.


Evolution of scale factor
scale factor
time (s)
10^{0}
10^{-5}
10^{-10}
10^{-15}
10^{-20}
10^{-25}
10^{-30}
10^{-35}
10^{-40}
10^{-45}
10^{-50}
10^{-55}
10^{-60}
10^{-45} 10^{-40} 10^{-35} 10^{-30} 10^{-25} 10^{-20} 10^{-15} 10^{-10} 10^{-5} 10^{0} 10^{5} 10^{10} 10^{15}

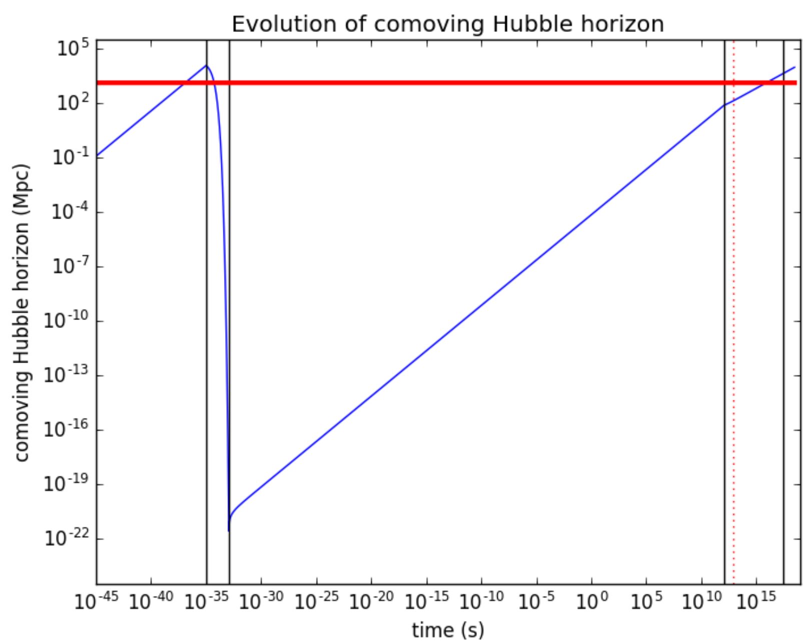

Evolution of comoving Hubble horizon
comoving Hubble horizon (Mpc)
time (s)
10^{5}
10^{2}
10^{-1}
10^{-4}
10^{-7}
10^{-10}
10^{-13}
10^{-16}
10^{-19}
10^{-22}
10^{-45} 10^{-40} 10^{-35} 10^{-30} 10^{-25} 10^{-20} 10^{-15} 10^{-10} 10^{-5} 10^{0} 10^{5} 10^{10} 10^{15}

Figure 1:

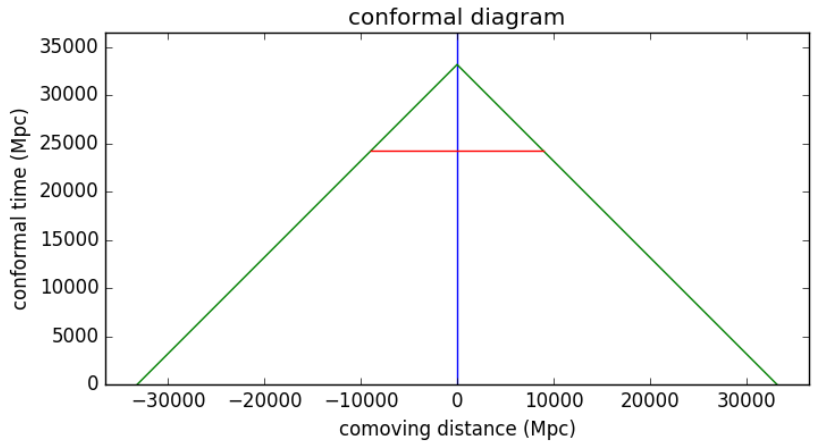


Figure 2:

Solution of problem 2

In the following we call $T_{\rm MeV}$ the temperature of the thermal soup in units of MeV. To compute the timescale for interaction of one electron and one neutrino we use the electron number density ($\mathcal{N}_s = 2$):

$$n_e = \frac{\zeta(3)}{\pi^2}\frac{3}{2}\left(\frac{k_B T}{\hbar c}\right)^3 = n_{e,\rm MeV}\, T_{\rm MeV}^3$$

where $n_{e,\rm MeV} = 2.35 \times 10^{31}$ cm^{-3}. Then:

$$\tau_w = \frac{1}{\sigma_w n_e c} = \tau_{\rm MeV}\, T_{\rm MeV}^{-5}$$

where $\tau_{\rm MeV} = 0.54$ s. To compute the age of the universe we need the total mass/energy density:

$$\rho = \frac{1}{c^2}\pi^2 30 g^\star \frac{(k_B T)^4}{(\hbar c)^3}$$

Then:

$$t = \sqrt{\frac{90 c^2 (\hbar c)^3}{32\pi^3 G (1\ {\rm MeV})^4}}\, g^{\star\ -1/2}\, T_{\rm MeV}^{-2} = t_{\rm MeV}\, g^{\star\ -1/2}\, T_{\rm MeV}^{-2}$$

where $t_{\rm MeV} = 2.4$ s. The equality $\tau = t$ leads to:

$$T_{\rm MeV} = \left(\frac{\tau_{\rm MeV}}{t_{\rm MeV}}\right)^{1/3} g^{\star\ 1/6} = 0.9\ {\rm MeV}\ \left(\frac{g^\star}{10.75}\right)^{1/6}$$

The neutron-to-baryon fraction at freezing time is:

$$X_{n,\rm fr} = \frac{1}{1 + {\rm e}^{1.3/T_{\rm MeV}}} = 0.19 \text{ if } N_\nu = 3$$

The Helium fraction will be roughly twice this fraction, if most neutrons end up in ^4He nuclei. This will be lowered from 0.38 to the observed value by the number of neutrons lost by beta decay before the opening of deuterium bottleneck. But the fraction of lost baryons will be the same if the time of opening of deuterium bottleneck does not depend on N_ν. The relative increase or decrease of neutrons when N_ν changes can thus be computed on the basis of $X_{n,\mathrm{fr}}$.

Repeating the computation for $N_\nu = 2$ or 4 we obtain respectively a decrease of 3.5% for 2 neutrinos and an increase of 2.9% for 4 neutrinos.

This simplistic approach gives a slight underestimate of the true variation of Y, that amounts to $\sim 5\%$; one of the reasons for this underestimate is that the opening of the deuterium bottleneck depends on the number of neutrinos, as discussed by Bonometto textbook (Sec. VI, page 168).