Cosmology 1

2020/2021
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Third intermediate test
Topic: early universe.

This is an extended version of the proposed problem of lecture 21, and a variation of the third intermediate test of 2019.

Consider a flat model with $H_0 = 67.4$ km s^{-1}, and ignore the contribution of Ω_Λ. Starting from the Big Bang, the scale factor evolves as follows: (1) from $t = 0$ to $t = t_i = 10^{-35}$ s it evolves like a radiation-dominated universe, (2) at t_i an exponential (de Sitter) inflationary phase starts with Hubble parameter H_i, lasting for N_e e-folds up to $t = t_f$ (so that $H(t_f - t_i) = N_e$), (3) from t_f to $t_{\rm eq}$, the time of equivalence, it evolves again like a radiation-dominated universe, (4) from $t_{\rm eq}$ to the present time t_0 it evolves like a matter dominated universe. We know that the equivalence redshift is $z_{\rm eq} = 3400$, and compute t_0 from the Hubble constant H_0 for an Einstein-De Sitter model (it will not be correct, but it is a tiny difference for our purposes). In this evolution the scale factor $a(t)$ and the Hubble parameter $H(t)$ remain continuous.

Find a (piecewise) analytic description of the scale factor $a(t)$ for the whole evolution of this universe, then compute the dimension of the comoving Hubble horizon $d_{cH} = c/\dot{a}$ (in units of Mpc) at $t = t_i$ and at $t = t_0$, and find the minimum number $N_{e,min}$ of e-folds needed to (barely) solve the horizon problem. Assume now that inflation lasted $N_e = N_{e,min} + 2$ e-foldings. Do not report all the calculations because you have the solution, just mention coincisely the various steps you have taken. Compute and report in a plot the following quantities, as a function of log cosmic time:

1. the comoving Hubble horizon;
2. the visible horizon today, computed for this Einstein-De Sitter model;
3. a scale that subtends 10 deg on the sky.

A colleague that has not studied inflation looks at your plot and is curious to know what it is; explain him/her in a few words the cosmic history of this "scale" (point 3 above) and why this plot solves the horizon problem. Plot the conformal past light cone to better clarify the point: how would you report the "scale" in that plot?

Solution

The full solution is given in the solution of the proposed problem 21 and in the solution of the Third Intermediate Test of 2019. I report here the latter.

The scale factor evolves in four phases:

$$a_1(t) = a_i \left(\frac{t}{t_i}\right)^{1/2}$$

$$a_2(t) = a_i e^{H(t-t_i)}$$

$$a_3(t) = a_f \left(\frac{t-t_3}{t_f-t_3}\right)^{1/2}$$

$$a_4(t) = a_{\rm eq} \left(\frac{t-t_4}{t_{\rm eq}-t_4}\right)^{2/3}$$

Here we have supposed that the extrapolations of $a(t)$ in phases 3 and 4 go to zero at times t_3 and t_4. The known data are H_0, t_i, $a_{\rm eq} = 1/(1+z_{\rm eq})$ and of course $a_0 = a(t_0) = 1$. We require that the four functions $a(t)$ and $H(t) = \dot{a}/a$ are continuous at times t_i, t_f and $t_{\rm eq}$, we set $a(t_0) = 0$ and $H(t_0) = H_0$, and we exploit the fact that $t_{\rm eq} \gg t_f$ and $t_0 \gg t_{\rm eq}$. We then obtain, as a function of a_i and N_e:

$$a_f = a_i e^N e$$

$$t_f = (2N_e + 1)t_i$$

$$t_3 = 2N_e t_i$$

$$t_{\rm eq} = t_i \left(\frac{a_{\rm eq}}{a_i e^{N_e}}\right)^2$$

$$t_4 = -\frac{1}{3} t_{\rm eq}$$

$$t_0 = \frac{2}{3H_0}$$

The (known) value for $a_{\rm eq}$ is obtained from $a_3(t_{\rm eq})$ and $a_4(t_{\rm eq})$ (assuming $a_4(t_0) = 1$), and this allows to find a_i:

$$a_i = e^{-N_e} a_{\rm eq}^{1/4} \left(\frac{4t_i}{3t_0}\right)^{1/2}$$

The comoving Hubble horizon for the four phases is:

$$d_{\rm cH1} = \frac{2ct_i}{a_i} \left(\frac{t}{t_i}\right)^{1/2}$$

$$d_{\rm cH2} = \frac{2ct_i}{a_i} e^{-H(t-t_i)}$$

$$d_{\rm cH3} = \frac{2ct_i}{a_i} e^{-N_e} \left(\frac{a(t)}{a_f}\right)$$

$$d_{\rm cH4} = \frac{2ct_i}{a_i} e^{-N_e} \left(\frac{a_{\rm eq}}{a_f}\right) \left(\frac{a(t)}{a_{\rm eq}}\right)^{1/2}$$

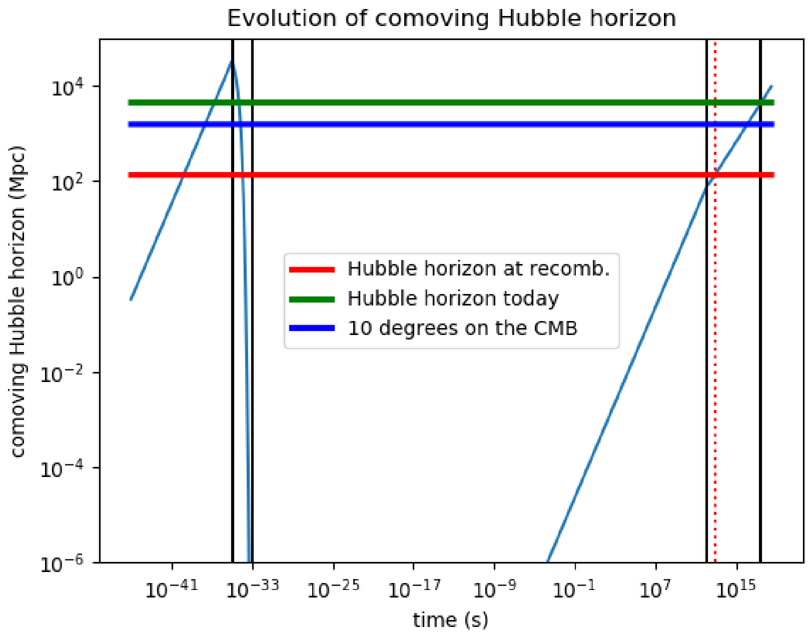

Figure 1:

As discussed in the lecture notes, the condition $d_{cH}(t = t_0) = d_{cHi} = 2ct_i/a_i$ gives:

$$e^{N_e} = \frac{a_{\rm eq}}{a_f}\left(\frac{1}{a_{\rm eq}}\right)^{1/2}$$

Using for a_f the expression given above, one obtains:

$$N_{e,min} = \frac{1}{4}\ln a_{\rm eq} + \frac{1}{2}\ln\frac{3t_0}{4t_i} = 58.2$$

For this $N_e = N_{e,min} + 2$ the size of the comoving Hubble horizon at the end of the four phases is: $d_{cH1} = 32900$ Mpc, $d_{cH2} = 2.25 \times 10^{-22}$ Mpc, $d_{cH3} = 76$ Mpc, $d_{cH0} = 4450$ Mpc.

The evoution of the comoving Hubble horizon is shown in the figure; in this case a log-log plot is clearly necessary, here we focus on the higher values, for the lower values see the plots in the 2019 solution. Here the recombination time $t_{\rm rec}$ is obtained by computing the time at which $a_4(t_{\rm rec}) = a_{\rm rec} = 1/(1+z_{\rm rec})$; in this case we do not assume that $t_{\rm rec} \gg t_{\rm eq}$, obtaining $t_{\rm rec} = 7.98 \times 10^{12}$ s $= 2.5 \times 10^5$ yr. The horizon at the equivalence is 134 Mpc, and is reported as a red horizontal line, together with the horizon today (green line). The scale subtended by 10 degrees in a flat Universe can be written simply as:

$$d_{10} = d_c(z = 1100)\frac{10}{180}\pi = 1553 \text{ Mpc}$$

where $d_c(z = 1100)$ is the comoving distance to redshift 1100.