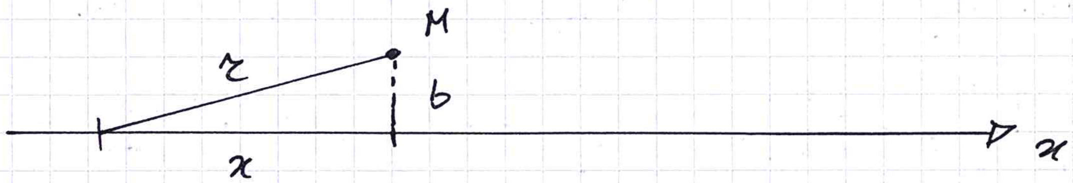
Notes on EHT

→ Photon trajectory in Newtonian dynamics:



$$x = ct \,, \quad r^2 = b^2 + x^2 = b^2 + c^2t^2 \Rightarrow 2r\frac{dr}{dt} = 2c^2t$$

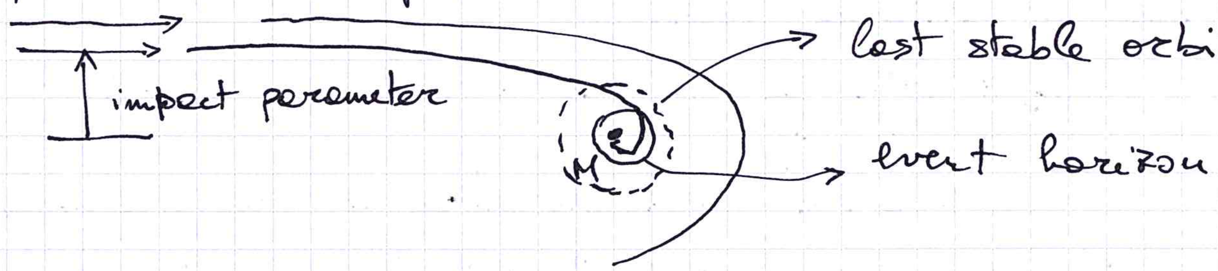
$$\Rightarrow \left(\frac{dr}{dt}\right)^2 = c^2\left(1 - \frac{b^2}{r^2}\right) \,, \quad V_{\text{eff}}(r) = \frac{c^2b^2}{r^2} = \frac{h^2}{r^2}$$

the straight path of the photon implies a centrifugal barrier

→ Impact parameter for a particle around a black hole:

at infinity $L = pb \Rightarrow b = \frac{L}{\sqrt{E^2 - m^2}}$, for a photon $b = \frac{L}{E}$

→ Capture radius of a black hole:



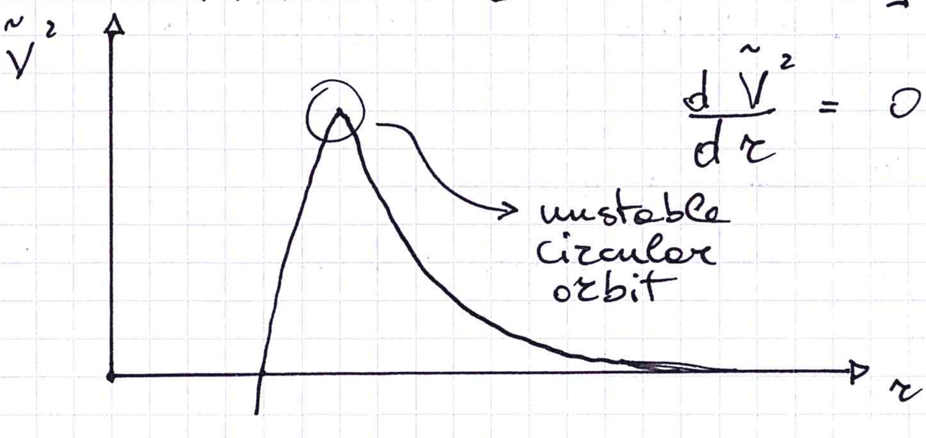
For a photon $(m = 0)$:

$$\left(\frac{dr}{d\lambda}\right)^2 = \tilde{E}^2 - \left(1 - \frac{2GM}{r}\right)\frac{\tilde{L}^2}{r^2} = \tilde{E}^2\left[1 - \left(1 - \frac{2GM}{r}\right)\frac{\tilde{L}^2}{\tilde{E}^2}\frac{1}{r^2}\right]$$

$$\tilde{E} \to E_\infty \,, \quad \frac{\tilde{L}}{\tilde{E}} = b$$

$$\left(\frac{dr}{d\lambda}\right)^2 = E_\infty\left[1 - \left(1 - \frac{2GM}{r}\right)\frac{b^2}{r^2}\right] \,, \quad \frac{\tilde{V}^2}{\tilde{E}^2} = \left(1 - \frac{2GM}{r}\right)\frac{b^2}{r^2}$$

$$\frac{d\tilde{V}^2}{dr} = 0 \Rightarrow r = 3GM$$



→ Photon capture radius:

The smallest radius for the photon orbit can be computed by setting

$$\frac{dr}{d\varphi} = \frac{dr}{d\lambda}\frac{d\lambda}{d\varphi} = 0 \quad , \quad \frac{d\varphi}{d\lambda} = \frac{\tilde{L}}{r^2} \quad \left(\tilde{L} = r^2 \frac{d\varphi}{d\lambda} \text{ is conserved}\right)$$

$$\left(\frac{dr}{d\varphi}\right)^2 = \frac{r^4}{b^2}\left[1 - \frac{b^2}{r^2}\left(1 - \frac{2GM}{r}\right)\right] = 0 \quad \text{for } b^2(r_{min} - 2GM) = r_{min}^3$$

This means that the smallest r for the orbit is the solution of this cubic equation.
But if $r_{min} = 3GM$ the photon falls onto the black hole.
This happens if $\quad b^2 < \frac{(3GM)^3}{(3GM - 2GM)} = 27\,(GM)^2$

This defines the photon capture radius:

$$b_{max} = \sqrt{27}\, R_G \qquad (R_G = GM \text{ , gravit. radius})$$

→ The shadow of the black hole is larger on the sky, due to extreme lensing

$$\vartheta_G \equiv \frac{GM}{c^2}\frac{1}{D} \quad , \quad d = \alpha\,\vartheta_G \quad , \quad \alpha \simeq 10$$

$$\alpha \simeq 2\sqrt{27} \quad \text{but geometry is more complex}$$

M87 is at $D = 16.8 \pm 0.8$ Mpc, the BH mass of M87* is $M = 6.5 \pm 0.7 \times 10^9\, M_\odot$, so:

$$R_G = \frac{GM}{c^2} = 8.85 \times 10^{14}\text{ cm} = 2.86 \times 10^{-4}\text{ pc}$$

$$\vartheta_G = \frac{R_G}{D} = 1.85 \times 10^{-11}\text{ rad} = 3.81\ \mu as$$

$$d \simeq 42\ \mu as \quad \text{and} \quad \frac{2\pi R_G}{c} \simeq 1.8 \times 10^5\,s \simeq 51\,h$$

→ For Sgr A*: $\quad D = 8$ kpc $\quad , \quad M = 4 \times 10^6\, M_\odot$.

$$R_G = 5.9 \times 10^{12}\text{ cm}$$

$$\vartheta_G = 2.39 \times 10^{-11}\text{ rad} = 4.92\ \mu as$$

$$d \simeq 50\ \mu as \quad , \quad \text{but} \quad \frac{2\pi R_G}{c} = 124\,s$$